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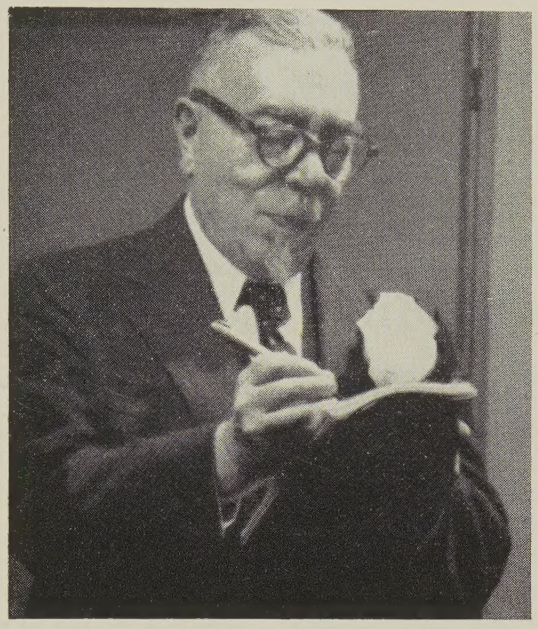

Norbert Wiener

Norbert Wiener was born in Columbia, Mo., on November 26, 1894. He received the A.B. degree from Tufts College in 1909, and did graduate work at Cornell and Harvard Universities, receiving from the latter the M.A. degree in 1912 and the Ph.D. degree in 1913. The following year he was awarded the John Thornton Kirkland Fellowship at Harvard which enabled him to study with Bertrand Russell in Cambridge, England and with Hilbert in Gottingen, Germany. He continued his studies with Russell and Hardy in England as Frederick Sheldon Fellow in 1915 and concluded the year with work at Columbia University.

After returning to this country, Dr. Wiener held various academic and commercial positions. He was Docent Lecturer in Harvard's Department of Philosophy for one year, and thereafter an instructor in Mathematics at the University of Maine. During 1917–1918 he worked for General Electric Corporation at Lynn, Mass. Before entering the army for a tour of duty at Aberdeen Proving Ground in Maryland, he was a staff writer for Encyclopedia Americana. In 1919 he worked briefly for the Boston Herald.

Since 1919, Dr. Wiener has been a faculty member of the Mathematics Department at the Massachusetts Institute of Technology, becoming a full Professor in 1932. During this time he has lectured extensively in the United States and abroad. In 1926 he was in Gottingen and Copenhagen under a Guggenheim Fellowship, in 1929 he became Exchange Professor of Mathematics at Brown University, and in 1931 spent one year as a lecturer at Cambridge University. His final position abroad before World War II was that of Visiting Professor at Tsing Hua University in Peiping, China.

In 1947, Dr. Wiener collaborated with Dr. Arturo Rosenblueth at the National Institute of Cardiology in Mexico. He lectured in the College de France of the University of Paris as a Fulbright Teaching Fellow in 1951, and during 1955–1956 was a visiting Professor at the Indian Statistical Institute in Calcutta.

Numerous prizes have been awarded to Dr. Wiener during his notable career, among them the Bowdoin Prize from Harvard in 1914, the Bocher Prize of the American Mathematical Society in 1933, and the Lord and Taylor American Design Award in 1949. Tufts College and the University of Mexico awarded him honorary Sc.D. degrees in 1946 and 1951 respectively.

He is a member of the American Mathematical Society and the London Mathematical Society.

What is Information Theory?

NORBERT WIENER

INFORMATION THEORY has been identified in the public mind to denote the theory of information by bits, as developed by Claude E. Shannon and myself. This notion is certainly important and has proved profitable as a standpoint at least, although as Dr. Shannon suggests in his editorial, "The Bandwagon," the concept as taken from this point of view is beginning to suffer from the indiscriminate way in which it has been taken as a solution of all informational problems, a sort of magic key. I am pleading in this editorial that Information Theory go back of its slogans and return to the point of view from which it originated: that of the general statistical concept of communication. A message is to be conceived as a sequence of occurrences distributed in time to be considered not exclusively by itself, but as one of an ensemble of similar sequences. As such it comes under the theory of time series which is an important branch of statistical theory with a rapidly developing technique and set of concepts of its own. This theory is closely allied to the ideas of Willard Gibbs in statistical mechanics. What I am urging is a return to the concepts of this theory in its entirety rather than the exaltation of one particular concept of this group, the concept of the measure of information into the single dominant idea of all.

I am pleading for this more particularly because the Gibbsian point of view is showing an applicability and fertility in many branches of science other than communication theory and in my opinion in all branches of science whatever. It is generally recognized that the quantum theory which now dominates the whole of physics is at root a statistical theory; although it is perhaps not yet as generally recognized as it should be, the quantum theory is strictly a branch of the theory of time series. Professor Armand Siegel and I are among those now working in this field.

What I am here entreating is that communication theory be studied as one item in an entire context of related theories of a statistical nature, and that it should not lose its integrity by becoming a special vested interest attached to a certain set of slogans and clichés. I hope that these TRANSACTIONS may encourage this integrated view of communication theory by extending its hospitality to papers which, while they bear on communication theory, cross its boundaries, and have a scope covering the related statistical theories. In my opinion we are in a dangerous age of overspecialization. To me the danger of this period is not primarily that we are studying very special problems that the development of science has forced us to go into, but rather that we are in great danger of finding our outlook so limited that we may fail to see the bearing of important ideas because they have been formulated in what our organization of science has decreed to be alien territory. I hope that these TRANSACTIONS may steadily set their face against this comminution of the intellect.

Optimum, Linear, Discrete Filtering of Signals Containing a Nonrandom Component*

KENT R. JOHNSON†

Summary—The problem of filtering nonrandom signals from tationary random noise has recently received considerable atten- ion. The filter design procedure developed by Wiener is not pplicable in this case since that procedure is predicted on the ssumption that the signal to be filtered is stationary and random. ately, both Booton and the team of Zadeh and Ragazzini have eveloped optimum filters for the smoothing of nonrandom signals; owever, both of these filters are of the continuous type, whereas n many applications in which discontinuous control is used there s need for discrete filters for such signals. This paper presents quations governing the design of a discrete version of the Zadeh- agazzini filter. The input signal is assumed to be the sum of a onrandom polynomial and a stationary random component and is ssumed to be obscured by stationary random noise.

An approximate formula for the output noise power of an optimum lter designed to make a zero-lag estimate of either its input unction or one of the derivatives thereof is derived for the important pecial case in which the noise is white and the signal is a non- andom polynomial. A brief discussion is given of the use of the lter with nonrandom, nonpolynomial signals.

Introduction

IN THE PAST few years there has been considerable interest in the design of filters for the separation of nonrandom signals from noise. Since a wide class of unctions can be approximated by polynomials it is often onvenient to assume that the nonrandom signal to be eceived is such a function. A suitable degree for the olynomial, and the time interval over which the approxi- nation will be valid, can be estimated from *a priori* nowledge of the quantity to be measured.

Recently Booton[1] has presented equations governing he design of optimum, time varying, continuous, linear ilters which may be used with signals containing non- andom as well as random components. Using his pro- edure it is possible to design a filter to yield an optimum stimate of the result of any desired time varying linear peration on the input. Unfortunately Booton's article oes not contain a general solution to the equations eveloped and does not give any estimate of the filter's utput noise power.

Zadeh and Ragazzini[2] have developed the optimum, ontinuous, time invariant, linear filter, having specified emory time, for use with signals consisting of both a onrandom polynomial and a stationary random com- onent. They assume the signal to be obscured by station- ry random noise. Their filter, like Booton's, differs from the Wiener filter[3] in that that filter is designed to operate on a stationary random signal only. The Zadeh-Ragazzini filter reduces to a Wiener filter in the case when the nonrandom part of the input is zero and the memory time is infinite.

Both the Booton filter and the Zadeh-Ragazzini filter are continuous. Because, however, the use of discontinuous control is indicated in many applications (to achieve increased accuracy, for example), the development of discrete filters for use with nonrandom signals seems worthwhile. Accordingly this article is devoted to the development and preliminary investigation of a discrete analog of the Zadeh-Ragazzini filter. A set of linear equations will be derived whose solution is the set of weighting coefficients to be used in the filter. In addition, some approximate formulas will be developed which give the ratio of output to input noise power for the case of optimum (least square sense), linear, zero-lag, unbiased, discrete filters having specified memory time and yielding as output an estimate of either the input signal or a derivative thereof. The formulas are valid for the special case in which the input is a polynomial and the obscuring noise is white.[4] The term "zero-lag" indicates that the filter estimates the value of the quantity of interest at the time of the latest sample.

Derivation of Filter Equations

Consider an ensemble of signals, each consisting of a polynomial and a stationary random component. Let each signal be obscured by stationary random noise. It should be understood that the polynomial, denoted by $P(t)$, is not random; *i.e.*, it remains the same over the entire ensemble. The polynomial's degree is known, but its coefficients are not. The random part of the signal, $M(t)$, and the random noise, $N(t)$, do, of course, vary over the ensemble. It will be assumed for convenience that both $M(t)$ and $N(t)$ have zero ensemble mean. The problem is to determine what linear combination of a fixed number of equally spaced samples of the input, (*i.e.*, signal plus noise), will give the optimum, in the sense of the least ensemble mean squared error, zero-bias estimate of the desired output function.

The entire signal may be written as

$$S(t) = P(t) + M(t) \tag{1}$$

* The work in this paper was done at the Ramo-Woolridge Corp. n Los Angeles, Calif.

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[1] R. C. Booton, Jr., "An optimization theory for time-varying near systems with nonstationary statistical inputs," Proc. IRE, ol. 40, pp. 977–981; August, 1952.

[2] L. A. Zadeh and J. R. Ragazzini, "An extension of Wiener's heory of proediction," *J. Appl. Phys.*, vol. 21, pp. 645–655; July, 950.

[3] Norbert Wiener, "Extrapolation, Interpolation, and Smoothing of Stationary Time Series," John Wiley and Sons, Inc., New York, N.Y.; 1950.

[4] That is, it is assumed that the correlation time of the noise is small compared to the interval at which the input is sampled.

and the input to the filter as

$$e_1(t) = S(t) + N(t). \tag{2}$$

The output of the filter is then

$$e_2(t) = \sum_{n=0}^{N} w_n e_1(t - nh) \tag{3}$$

where the w_n are constants yet to be determined. It is assumed that the sampling interval, denoted by h, is constant. Let the desired filter output be

$$S^*(t) = AS(t) \tag{4}$$

where A is an operator which may be represented as a linear combination of derivatives. Define the output error, ϵ, by

$$\epsilon = S^*(t) - e_2(t) \tag{5}$$

and stipulate that

$$\bar{\epsilon} = \overline{AS(t)} - \overline{e_2(t)} = AP(t) - \sum_{n=0}^{N} w_n P(t - nh) = 0, \tag{6}$$

so that the estimate of $S^*(t)$ is unbiased. An unbiased estimate is required because it seems desirable to have zero output error when there is no input noise. This seems a reasonable requirement since in application the input noise level may vary considerably. The bar in (6) indicates an ensemble average. In obtaining this equation the zero mean requirement on $M(t)$ and $N(t)$ has been used.

The polynomial $P(t - nh)$ may be written as

$$P(t - nh) = \sum_{i=0}^{k} \frac{(-nh)^i}{j!} P^{(i)}(t), \tag{7}$$

where $P^{(i)}(t)$ is the jth derivative of $P(t)$ and k is the degree of the polynomial. Substituting (7) into (6) yields

$$AP(t) = \sum_{i=0}^{k} \sum_{n=0}^{N} w_n \frac{(-nh)^i}{j!} P^{(i)}(t) = \sum_{i=0}^{k} \frac{(-h)^i \mu_i}{j!} P^{(i)}(t), \tag{8}$$

in which

$$\mu_i = \sum_{n=0}^{N} w_n n^i \tag{9}$$

is the jth moment of the weighting coefficients w_n. If $AP(t)$ is now written as a Taylor's Series, the two sides of (8) may be compared term by term to yield $k + 1$ constraints, in the form of the required values of the moments μ_i, on the weighting coefficients, w_n. Thus, if A is the identity operator

$$\mu_0 = 1$$

$$\mu_i = 0 \qquad j \neq 0.$$

If A is the first differential operator,

$$\mu_1 = -\frac{1}{h}$$

$$\mu_i = 0 \qquad j \neq 1.$$

By using (6) for $AP(t)$ it is now possible to write the output noise, ϵ, as

$$\epsilon = AM(t) - \sum_{n=0}^{N} w_n[M(t - nh) + N(t - nh)]. \tag{10}$$

The ensemble average of the noise power then becomes

$$\bar{\epsilon}^2 = \overline{[AM(t)]^2} - 2 \sum_{n=0}^{N} w_n \psi_A(n) + \sum_{n=0}^{N} \sum_{i=0}^{N} w_n w_i [\psi_M(n, i) + \psi_N(n, i)], \tag{11}$$

where

$$\psi_A(n) = \overline{[AM(t)]M(t - nh)} \tag{12}$$

$$\psi_M(n, i) = \overline{M(t - nh)M(t - ih)}, \tag{13}$$

and

$$\psi_N(n, i) = \overline{N(t - nh)N(t - ih)}. \tag{14}$$

In obtaining (11), $M(t)$ and $N(t)$ have been assumed uncorrelated.

It is now desired to minimize $\bar{\epsilon}^2$ subject to the $k +$ constraints expressed by (8). This may be done by the usual method of Lagrange multipliers. Consider the expression

$$I = \bar{\epsilon}^2(w_n) - \sum_{j=0}^{k} \lambda_j \sum_{n=0}^{N} w_n n^j. \tag{15}$$

The λ_j are constants. In order for a set of w_n to minimize $\bar{\epsilon}^2$ subject to the constraints previously mentioned, it is necessary that there exist a set of λ_j such that the quantity I is rendered stationary by the aforementioned set of w. That is, it is necessary to find a set of w_n and a set of λ such that all the equations

$$\frac{\partial I}{\partial w_n} = 0 \qquad n = 0 \cdots N \tag{16a}$$

and

$$\sum_{n=0}^{N} w_n n^j = \mu_j \qquad j = 0 \cdots k \tag{16b}$$

are satisfied. These are $N + k + 2$ equations in $N + k +$ unknowns. In the case at hand they are linear in the w and the λ_j. Eqs. (16) may be written as

$$2 \sum_{i=0}^{N} w_i \psi(n, i) - \sum_{j=0}^{k} \lambda_j n^j = 2\psi_A(n) \quad n = 0 \cdots N \tag{17a}$$

and

$$\sum_{n=0}^{N} w_n n^j = \mu_j \qquad j = 0 \cdots k, \tag{17b}$$

where

$$\psi(n, i) = \psi_M(n, i) + \psi_N(n, i). \tag{18}$$

Eqs. (11) and (17a) may be combined to yield an expression for the minimum value of $\bar{\epsilon}^2$, denoted by ϵ. Thus,

$$\bar{\epsilon}_m^2 = \overline{[AM(t)]^2} - \sum_{n=0}^{N} w_n \psi_A(n) + \frac{1}{2} \sum_{j=0}^{k} \lambda_j \mu_j. \tag{19}$$

Eqs. (17) may be written in matrix form as

$$\left[\begin{array}{ccccc|cccccc} \psi(0,0) & \cdots\cdots & \psi(0,N) & & & 1 & 0 & 0 & \cdot\ \cdot & 0 \\ \cdot & & \cdot & & & 1 & 1 & 1 & \cdot\ \cdot & 1 \\ \cdot & & \cdot & & & 1 & 2 & 2^2 & \cdot\ \cdot & 2^k \\ \cdot & & \cdot & & & \cdot & & & & \cdot \\ \cdot & & \cdot & & & \cdot & & & & \cdot \\ \cdot & & \cdot & & & \cdot & & & & \cdot \\ \psi(N,0) & \cdots\cdots & \psi(N,N) & & & 1 & N & N^2 & \cdot\ \cdot & N^k \\ \hline 1 & 1 & 1 & \cdots\cdot & 1 & & & & & \\ 0 & 1 & 2 & & N & & & & & \\ 0 & 1 & 2^2 & & N^2 & & & & & \\ \cdot & \cdot & \cdot & & \cdot & & & 0 & & \\ \cdot & \cdot & \cdot & & \cdot & & & & & \\ 0 & 1 & 2^k & \cdots\cdot & N^k & & & & & \end{array}\right]\left[\begin{array}{c} w_0 \\ w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_N \\ \hline -\frac{\lambda_0}{2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -\frac{\lambda_k}{2} \end{array}\right] = \left[\begin{array}{c} \psi_A(0) \\ \psi_A(1) \\ \psi_A(2) \\ \cdot \\ \cdot \\ \cdot \\ \psi_A(N) \\ \hline \mu_0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mu_k \end{array}\right] \tag{20a}$$

or

$$\begin{bmatrix} \psi & B \\ B' & 0 \end{bmatrix}\begin{bmatrix} J \\ R \end{bmatrix} = \begin{bmatrix} P \\ Q \end{bmatrix} \tag{20b}$$

where the submatrices in (20b) are as shown above. The prime on B indicates its transpose. With the matrix notation the error $\overline{\epsilon_m^2}$ becomes

$$\overline{\epsilon_m^2} = \overline{[AM(t)]^2} - J'(BR + P), \tag{21}$$

in which each side of the equation is a matrix with one element. Eq. (20b) may be inverted to yield

$$\begin{bmatrix} J \\ R \end{bmatrix} = \begin{bmatrix} E & F \\ F' & H \end{bmatrix}\begin{bmatrix} P \\ Q \end{bmatrix} \tag{22}$$

where

$$B = \Psi^{-1} - \Psi^{-1}B(B'\Psi^{-1}B)^{-1}B'\Psi^{-1} \tag{23}$$

$$F = \Psi^{-1}B(B'\Psi^{-1}B)^{-1} \tag{24}$$

$$N = -(B'\Psi^{-1}B)^{-1}. \tag{25}$$

Thus the weighting coefficients are given by

$$J = EP + FQ. \tag{26}$$

The actual matrix inversions indicated in (23), (24), and (25) are in general quite tedious.

If the signal consists of a polynomial alone: *i.e.*, $M = 0$, (22) through (25) may be combined with (21) to yield a compact expression for the ratio $\overline{\epsilon_m^2}$ to p, the total input noise power.

$$\frac{\overline{\epsilon_m^2}}{p} = \frac{Q'(B'\Psi_N^{-1}B)^{-1}Q}{p} = Q'(B'D^{-1}B)^{-1}Q. \tag{27}$$

Here

$$\Psi_N = \begin{matrix} \psi_N(0,0) & \cdots & \psi_N(0,N) \\ \vdots & & \\ \psi_N(N,0) & \cdots & \psi_N(N,N) \end{matrix} \tag{28}$$

and

$$D = \frac{\Psi_N}{p}, \quad \text{with} \quad p = \psi_N(0,0). \tag{29}$$

The matrix D depends only on the shape of the correlation function $\psi_N(i, n)$; it is independent of the total noise power p. Thus (27) indicates that when the signal is a polynomial the ratio $\overline{\epsilon_m^2}/p$ is independent of input noise power; *i.e.*, the output noise power is a linear function of input noise power.

Because the number of equations, $(N + k + 2)$, which determine the weighting coefficients is large, the use of a digital computer may well be indicated for their solution. It is possible, however, for an interesting special case, to obtain an approximate formula for the noise power output of the filter without actually determining the weighting coefficients. This formula is presented below.

$$\frac{\overline{\epsilon_m^2}}{p} = \frac{1}{(hN)^{2i}N}\left[\frac{(k+j+1)!}{(k-j)!j!}\right]^2 \frac{1}{2j+1}. \tag{30}$$

The derivation of (30) has been relegated to an Appendix because of its length. There it is shown that the above result is valid under the following conditions.

1) The signal is a pure polynomial; *i.e.*, $M(t) = 0$.
2) The correlation time of the noise is small compared to the sampling interval.
3) The number of samples used in the filter is large compared to the degree of the polynomial on which the filter is designed to operate, $(N >> k)$.

Notice that in (30) the product hN is the filter memory time T. Thus, for a given number of samples, and a given degree of polynomial, the ratio of rms, output noise to rms, input noise will vary inversely as T^j for a jth order differentiating filter. This behavior could have been predicted from an examination of less elaborate differentiating discrete filters. The inverse variation of the above mentioned ratio with the square root of the number of samples, for fixed memory time, may be interpreted loosely as the usual improvement in accuracy which results from making repeated measurements of the same quantity.

Filtering Nonrandom, Nonpolynomial Signals

The point in going to some length to develop a device which is capable of filtering a nonrandom polynomial from noise lies, of course, in the fact that many nonrandom signals of interest may be approximated by polynomials over lengths of time of sufficient duration to allow appreciable discrimination against noise. Questions which immediately arise in designing a filter for a specific application are, what degree should be assumed for the polynomial, and what memory time should be used? It is at once evident that interacting factors influence the choice of these two quantities. Thus, for a given polynomial degree and sampling interval, the noise filtering improves as memory time is increased. On the other hand, as memory time increases, if the signal is not a true polynomial the degree of polynomial which is assumed to represent the signal must usually be increased. This increase in assumed polynomial degree decreases the capacity of the filter to discriminate against noise, as is revealed by an inspection of (30).

At first glance it might be thought that arbitrarily great noise discrimination could be achieved by designing a filter to operate on a polynomial of sufficiently high degree to adequately approximate the signal over a length of time sufficient to provide the desired noise filtering. Additional work which is not in a sufficient state of completion to warrant publication leads the author to feel that while this is true for certain classes of signals; *e.g.*, polynomial, it is not true in general. Proof of this has not yet been obtained, however. In practice it will usually be necessary to effect a compromise between good noise discrimination, with long memory time and attendant difficulties in curve fitting, and good curve fitting with short memory time and limited noise filtering.

Discussion

The work just presented contains sufficient information for the design of an optimum linear discrete filter to operate on a signal composed of a stationary random component and a nonrandom polynomial. An approximate expression for output noise power has been presented for the special case in which the signal consists of only a nonrandom polynomial and in which the correlation time of the noise is small compared to the sampling interval. Perhaps the most notable deficiency in the present treatment lies in the absence of any investigation of the filter's steady state transfer function. While some work has been done in this direction, it was not felt that the material was in a sufficient state of completion to warrant its presentation.

Appendix

Approximate Expression for Output Noise Power

In the case when the signal consists of a pure polynomial; *i.e.*, $M(t) = 0$, and when the operator A is either the identity operator or a derivative operator, the expression for output noise power given in (19) reduces to a single term. Thus

$$\overline{\epsilon_m^2} = \tfrac{1}{2}\lambda_j\mu_j. \qquad \text{(No summation on } j\text{)}. \tag{31}$$

Here j is the order of the derivative represented by the operator A. If A is the identity operator, j is zero. The value of μ_j is determined by A as was described previously; thus, it is only necessary to find λ_j in order to determine the output noise power. The problem will be simplified by assuming that the correlation time of the noise is small compared to the sampling interval. This assumption causes the matrix ψ to become diagonal. For ease in computation the noise power, $p = \psi_N(0, 0)$, will be taken to be unity. Since the output noise power is a linear function of the input noise power under the condition of the case being considered, this does not constitute any restriction on the generality of the result.

Under the assumption made in the preceding paragraph (20a) becomes

$$\left[\begin{array}{cccc:cccc}
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 1 & 1 & N & \cdots & N^k \\
\hdashline
1 & 1 & \cdots & 1 & & & & \\
0 & 1 & \cdots & N & & & & \\
\vdots & \vdots & & \vdots & & & 0 & \\
0 & 1 & \cdots & N^j & & & & \\
\vdots & \vdots & & \vdots & & & & \\
0 & 1 & \cdots & N^k & & & &
\end{array}\right]
\left[\begin{array}{c} w_0 \\ w_1 \\ \vdots \\ w_N \\ \hdashline -\frac{\lambda_0}{2} \\ -\frac{\lambda_1}{2} \\ \vdots \\ -\frac{\lambda_j}{2} \\ \vdots \\ -\frac{\lambda_k}{2} \end{array}\right]
=
\left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \hdashline 0 \\ 0 \\ \vdots \\ \mu_j \\ \vdots \\ 0 \end{array}\right] \cdot \tag{32}$$

An application of Crammer's rule yields

$$-\frac{\lambda_j}{2} = \frac{\mu_j M_{N+j+2,N+j+2}}{M} \tag{33}$$

where M is the determinant of the square matrix in (32) and $M_{N+j+2,N+j+2}$ is the minor of its $N+j+2, N+j+$

lement. Both of these determinants may be simplified y the following procedure.

1) Add -1 times the elements of the first column to the elements of the $N + 2$ column.
2) Add -1 times the elements of the second column to the elements of the $N + 2$ through the $N + k + 2$ column.
3) Add $(-2)^i$ times the elements of the third column to the elements of the $N + j + 2$ column, where j goes from 0 through k.
4) Continue the process just outlined until the elements of the submatrix B are all replaced by zeroes.

The resulting determinant can immediately be reduced n order from $N + k + 2$ to $k + 1$. Thus

$$M = (-1)^{k+1} \begin{vmatrix} N+1 & \sum_{p=1}^{N} p & \cdots & \sum_{p+1}^{N} p^k \\ \sum_{p=1}^{N} p & \sum_{p=1}^{N} p^2 & \cdots & \sum_{p=1}^{N} p^{k+1} \\ \vdots & \vdots & & \vdots \\ \sum_{p=1}^{N} p^k & \sum_{p=1}^{N} p^{k+1} & \cdots & \sum_{p=1}^{N} p^{2k} \end{vmatrix}. \quad (34)$$

Similarly the determinant $M_{N+j+2,N+j+2}$ may be reduced to

$$M_{N+j+2,N+j+2} = (-1)^k \begin{vmatrix} N+1 & \sum_{p=1}^{N} p & \cdots & \sum_{p=1}^{N} p^{j-1} & \sum_{p=1}^{N} p^{j+1} & \cdots & \sum_{p=1}^{N} p^k \\ \sum_{p=1}^{N} p & \sum_{p=1}^{N} p^2 & \cdots & \sum_{p=1}^{N} p^j & \sum_{p=1}^{N} p^{j+2} & \cdots & \sum_{p=1}^{N} p^{k+1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \sum_{p=1}^{N} p^{j-1} & \sum_{p=1}^{N} p^j & \cdots & \sum_{p=1}^{N} p^{2j-2} & \sum_{p=1}^{N} p^{2j} & \cdots & \sum_{p=1}^{N} p^{k+j-1} \\ \sum_{p=1}^{N} p^{j+1} & \sum_{p=1}^{N} p^{j+2} & \cdots & \sum_{p=1}^{N} p^{2j} & \sum_{p=1}^{N} p^{2j+2} & \cdots & \sum_{p=1}^{N} p^{k+j+1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \sum_{p=1}^{N} p^k & \sum_{p=1}^{N} p^{k+1} & \cdots & \sum_{p=1}^{N} p^{k+j-1} & \sum_{p=1}^{N} p^{k+j+1} & \cdots & \sum_{p=1}^{N} p^{2k} \end{vmatrix} \quad (35)$$

For low values of k it is possible to evaluate λ_i immediately. If this is done for $k = 1$ and 2 and the result used in (31), the following expressions for noise power are obtained.

$= 1$

$$\overline{\epsilon_m^2} = \frac{12}{Nh^2(N+1)(N+2)}, \qquad j = 1 \quad (36)$$

$= 2$

$$\overline{\epsilon_m^2} = \frac{12(2N+1)(8N-3)}{Nh^2(N^2+1)(N+2)(N+3)}, \qquad j = 1. \quad (37)$$

Similarly, for $k = 2$ and $j = 2$,

$$\overline{\epsilon_m^2} = \frac{720}{Nh^4(N^2+1)(N+2)(N+3)}. \quad (38)$$

Eq. (36) is the output noise power of an $N + 1$ sample, zero lag, measurement of velocity when displacement is a first degree polynomial. Eqs. (37) and (38) are the corresponding expressions for zero lag measurements of velocity and acceleration, respectively, when displacement is a second degree polynomial. In all cases, of course, the input noise power is taken to be unity.

The direct evaluation of the determinants of (34) and (35) becomes quite tedious when k is large. To further simplify the problem of finding output noise, it is possible to utilize the fact that often the solutions of greatest interest are those for large values of N. In order to effect the simplification so afforded, it will be necessary to digress for a bit on the evaluation of sums of the form

$$S_n(N) = \sum_{n=0}^{N} p^n. \quad (39)$$

Let

$$p^{(n)} = p(p-1)(p-2)\cdots(p-n+1), \quad (40)$$

where n is a positive integer. Then

$$\begin{aligned} p^{(1)} &= p \\ p^{(2)} &= p(p-1) = p^2 - p \\ p^{(3)} &= p(p-1)(p-2) = p^3 - 3p^2 + 2p \\ &\text{etc.} \end{aligned} \quad (41)$$

It is easy to see from (41) that it is possible to express p^n in terms of a summation of the form.

$$p^n = p^{(n)} + a_1 p^{(n-1)} + a_2 p^{(n-2)} + \cdots + a_{n-1} p^{(1)}. \quad (42)$$

Notice that the coefficient of $p^{(n)}$ will always be unity. The reason for putting the p^n in terms of the $p^{(n)}$ is that it is very simple to sum terms in $p^{(n)}$ over integer values

of p. Thus

$$\sum_{p=1}^{N} p^{(n)} = \frac{p^{(n+1)}}{n+1}\Big|_{p=1}^{p=N+1} = \frac{(N+1)^{(n+1)} - 1^{(n+1)}}{n+1}$$

$$= \frac{(N+1)^{(n+1)}}{n+1}. \tag{43}$$

The relation given in (43) is an elementary result in the calculus of finite differences. It is now apparent that by using (42) and (43) it is possible to write a summation of the type appearing in (39) as a polynomial in N of degree $n+1$. Thus every element in the determinant M will be a polynomial whose degree is one greater than the power to which p is raised in the summation which the polynomial represents. Furthermore, from (42) and (43), it is apparent that in each polynomial the coefficient of the highest power of N will be the reciprocal of that power.

It is possible to write M in the form

$$M = e^{i_1 i_2 \cdots i_{k+1}} b_{i_1}^1 b_{i_2}^2 \cdots b_{i_{k+1}}^{k+1}, \tag{44}$$

where a repeated index indicates a summation. The factor b_m^j is the element in the jth row and the mth column of M. The quantity

$$e^{i_1 i_2 \cdots i_{k+1}}$$

has the value zero if any two of the i_n are equal. If none of the indices are equal it has the value $+1$ or -1, depending on whether an even or odd number of interchanges of indices is required to put the indices in the order $1, 2, 3, \cdots k+1$. Since each element b_m^j is a polynomial of degree $j + m - 1$, it is apparent that each term in the summation of (44) will be a polynomial. Also, since the lower indices of the b_m^j in (44) must, in any term with a nonzero coefficient, be a permutation of the integers 1 through $k+1$, the degree, D, of each term must be

$$D = \sum_{j=1}^{k+1} (j-1) + \sum_{m=1}^{k+1} m$$

$$= 2\sum_{j=1}^{k+1} j - (k+1) - (k+1)^2. \tag{45}$$

Thus it is established that each term of (44) will be a polynomial of degree $(k+1)^2$. From this it follows that M itself will be such a polynomial and that the coefficient of the $(k+1)^2$ power of N will be the sum of the coefficients of this power of N in the individual terms of (44). But in each term of (44) the coefficient of this power of N is just the product of the coefficients of the highest powers of N in the individual factors b_m^j appearing in that term. Thus, the coefficient of the $(k+1)^2$ power of N in the determinant M may be found by replacing in (44) each element b_m^j by the coefficient of the highest power of N appearing in that element. This has been shown to be $(j+m-1)^{-1}$. The resulting expression is just the expansion of the determinant.

$$Q_k = (-1)^{k+1} \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{k+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{k+3} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{k+1} & \frac{1}{k+2} & \frac{1}{k+3} & \cdots & \frac{1}{2k+1} \end{vmatrix} \tag{46}$$

In a similar fashion it is easy to show that the highest power of N in the determinant $M_{N+j+2,N+j+2}$ is $(k+1)^2 - 2j - 1$. Also, it can be shown that the coefficient of this term is

$$V_{k,j} = (-1)^k \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{j} & \frac{1}{j+2} & \cdots & \frac{1}{k+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{j+1} & \frac{1}{j+3} & \cdots & \frac{1}{k+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{j+2} & \frac{1}{j+4} & \cdots & \frac{1}{k+3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{1}{j} & \frac{1}{j+1} & \frac{1}{j+2} & \cdots & \frac{1}{2j-1} & \frac{1}{2j+1} & \cdots & \frac{1}{k+j} \\ \frac{1}{j+2} & \frac{1}{j+3} & \frac{1}{j+4} & \cdots & \frac{1}{2j+1} & \frac{1}{2j+3} & \cdots & \frac{1}{k+j+2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{1}{k+1} & \frac{1}{k+2} & \frac{1}{k+3} & \cdots & \frac{1}{k+j} & \frac{1}{k+j+2} & \cdots & \frac{1}{2k+1} \end{vmatrix} \tag{47}$$

Thus to obtain an approximation for λ_i which is valid r large values of N it is only necessary to substitute in e numerator and denominator of (33) the terms con- ining the highest power of N appearing in $M_{N+i+2,N+i+2}$, nd M, respectively. This yields

$$-\frac{\lambda_i}{2} = \mu_i \frac{V_{k,i}N^{(k+1)^2-2i-1}}{Q_k N^{(k+1)^2}} = \mu_i \frac{V_{k,i}}{Q_k N^{2i+1}}. \tag{48}$$

There remains the task of evaluating the ratio $V_{k,i}/Q_k$.[5] recurrence relation involving the determinant Q_k may e derived as follows.

1) Subtract the $k + 1$ column of Q_k from all other columns.
2) Factor all quantities common to an entire row or column.
3) Subtract the $k + 1$ row of Q_k from all other rows.
4) Repeat step 2.

follows immediately that

$$Q_k = -\frac{(k!)^4}{(2k+1)!(2k)!} Q_{k-1}. \tag{49}$$

n an entirely similar fashion

$$V_{k,i} = -\frac{(k!)^4(k+j+1)^2}{(2k+1)!(2k)!(k-j)^2} V_{k-1,i}. \tag{50}$$

hus

$$\frac{V_{k,i}}{Q_k} = \frac{(k+j+1)^2}{(k-j)^2} \frac{V_{k-1,i}}{Q_{k-1}}$$

$$= \frac{(k+j+1)^2 \cdots (2j+2)^2}{(k-j)^2 \cdots (1)^2} \frac{V_{i,i}}{Q_i}. \tag{51}$$

[5] The method used for this evaluation is due to Dr. G. J. Gleghorn the Ramo-Wooldridge Corporation.

But

$$V_{i,i} = Q_{i-1}, \tag{52}$$

so, combining (49), (51), and (52), the following result is obtained.

$$\frac{V_{k,i}}{Q_k} = -\left[\frac{(k+j+1)!}{(k-j)!}\right]^2 \frac{1}{(2j+1)(j!)^4}. \tag{53}$$

If (53) is substituted into (48) to yield $-\lambda_{i/2}$, and the result is substituted into (31) an expression for output noise power results.

$$\overline{\epsilon_m^2} = \frac{\mu_j^2}{N^{2j+1}} \left[\frac{(k+j+1)!}{(k-j)!}\right]^2 \frac{1}{(2j+1)(j!)^4}. \tag{54}$$

The input noise power is assumed to be unity. The moment μ_i may be evaluated from (8). When the operator A represents the jth derivative

$$\mu_i = \frac{j!}{h^j}(-1)^j. \tag{55}$$

So

$$\overline{\epsilon_m^2} = \frac{1}{(hN)^{2j}N} \left[\frac{(k+j+1)!}{(k-j)!j!}\right]^2 \frac{1}{2j+1}. \tag{56}$$

This is the result presented in (30).

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Spatial Filtering in Optics*

EDWARD L. O'NEILL†

Summary—Starting with the formulation of H. H. Hopkins for the image forming properties of an optical system in terms of a coherence factor over the object plane, the two extreme cases of complete coherence and incoherence are considered. The incoherent case is treated briefly as a low-pass spatial frequency filter.

In the case of coherent illumination, it is shown that the optical analog of such well-known electrical concepts as equalization [17], edge-sharpening, and the detection of periodic and isolated signals in the presence of noise can be carried out with relative ease. A detailed theoretical treatment of the problem together with illustrations emphasizes the analogy between optical and electrical filtering.

List of Symbols and Fourier Relations[1]

$\hat{t}(x_i, y_i)$ = point image amplitude distribution.
$t(x_i, y_i)$ = point image intensity distribution.
$\hat{o}(x_0, y_0)$ = complex amplitude transmission of object.
$o(x_0, y_0)$ = object intensity distribution.
$\hat{\imath}(x_i, y_i)$ = image amplitude distribution.
$i(x_i, y_i)$ = image intensity distribution.
$\gamma(x_0, y_0; x_0', y_0')$ = "partial coherence factor" in object plane.
$\hat{\tau}(\mu, \nu)$ = complex transmission of aperture.
$\tau(\mu, \nu)$ = transfer function.
$\hat{O}(\mu, \nu)$ = object amplitude spatial spectrum.
$O(\mu, \nu)$ = object intensity spatial spectrum.
$\hat{I}(\mu, \nu)$ = image amplitude spatial spectrum.
$I(\mu, \nu)$ = image intensity spatial spectrum.
$\Gamma(\mu, \nu)$ = intensity variation over the source.

All functions denoted by the same small and capital letter are Fourier Transform pairs, *e.g.*,

$$\hat{o}(x_0, y_0) = \iint_{-\infty}^{\infty} \hat{O}(\mu, \nu) e^{-i(\mu x_0 + \nu y_0)} \, d\mu \, d\nu.$$

Introduction

THERE has been sufficient evidence over the past few years to indicate that the concepts of electrical communication theory have become a permanent fixture in the field of optics [1], [2], [4], [6], [12], [21-23], so that a detailed description of the electrical-optical analog is superfluous at this point. Nevertheless, it can be pointed out, in retrospect, that most of the emphasis thus far has been confined to the passive role of determining the performance of the optical system [8], [14-16], [23-25], namely, the evaluation of the transfer function or sine wave response curve. A study of the exceptions to this statement shows that, even in these cases, strictly speaking, a communication theory approach has not been employed [10], [13].

With the publication of an article by Maréchal and Croce [17] on increasing the contrast of photographs, it became apparent that the more general problem of detecting and recognizing signals in the presence of noise might be approached through the concepts of electrical communication theory. Not only was the general synthesis of an optical system possible, but it appeared immediately to involve some simplifications as compared to its electrical counterpart.

Rather than consider each optical system as a specia problem, it was decided to treat the general problem of image formation by including a factor which describes the illumination over the object plane. For this reason the formulation of Hopkins [8], [9] was adopted to evaluate the final intensity distribution in terms of the "partia coherence factor." It was then possible to discuss the two extreme cases of completely coherent and incoherent systems. The former, being linear in amplitude, could be handled from the Fourier standpoint. The latter being linear in intensity, was also subject to Fourier analysis and synthesis along with certain basic limitations However, since in the coherent case, control over the Fourier components that make up the image could be exercised with relative ease, it was possible to carry out a series of simple experiments that bore a one-to-one correspondence to known filtering operations in electrical communication theory. The paper presents both a theoretical and experimental treatment of the general formulation of the problem together with illustrations of the results.

The Fourier Analysis and Synthesis of Coherent and Incoherent Optical Systems

Generalized Image Formation

In any comprehensive treatment of image formation in an optical system, it is convenient to incorporate a factor which describes the mode of illumination over the object plane. For monochromatic systems Hopkins [8], [9] has succeeded in doing this in terms of a "partia coherence factor." Because of the comprehensive characte of such a formulation, this term will be employed in the analysis to be presented in this paper. However, only the two extremes of complete incoherence and coherenc will be considered here.

As a starting point, consider an element of the source $d\sigma$ (Fig. 1) producing a complex disturbance $\hat{u}(x_0, y_0)$ a the point (x_0, y_0) in the object space where $\hat{u}(x_0, y_0)$ is

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[1] This notation follows that of Linfoot and Fellgett [11] in which the "hooked" (^) variables denote complex amplitude and the "unhooked" variables denote intensity.

olution of the scalar wave equation. The resultant omplex transmitted amplitude is now given by $\hat{u}(x_0, y_0)$ $x_0, y_0)$. To determine the image distribution $\hat{\imath}(x_i, y_i)$, is necessary to integrate the point image distribution $x_i, y_i)$ over the object plane in the following manner:

$$x_i, y_i) = \iint_{-\infty}^{\infty} \hat{t}(x_i - x_0, y_i - y_0)\hat{u}(x_0, y_0)\hat{o}(x_0, y_0)\, dx_0\, dy_0. \tag{1}$$

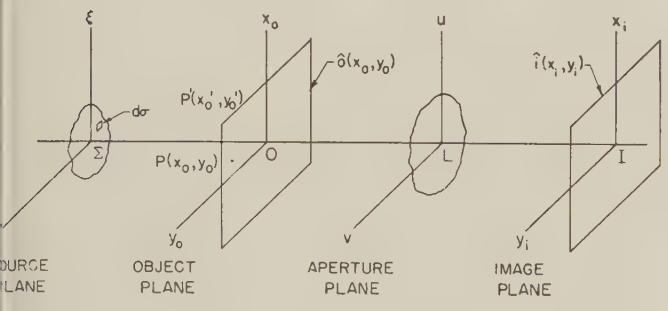


Fig. 1—Generalized image formation.

However, the eye or any physical instrument is sensitive o intensity variations, hence we must determine the nage distribution from an independent point in object pace (x_0', y_0') and perform a similar integration over the omplex conjugate of the functions in (1). We have then

$$di(x_i, y_i) = \hat{\imath}(x_i, y_i)\hat{\imath}^*(x_i, y_i)\, d\sigma \tag{2a}$$

r

$$i(x_i, y_i) = d\sigma \iiiint_{-\infty}^{\infty} \hat{t}(x_i - x_0, y_i - y_0)\hat{t}^*(x_i - x_0', y_i - y_0')$$
$$\cdot\hat{u}(x_0, y_0)\hat{u}^*(x_0', y_0')\hat{o}(x_0, y_0)\hat{o}^*(x_0', y_0')\, dS_0\, dS_0' \tag{2b}$$

here

$$dS_0 = dx_0\, dy_0 \quad \text{and} \quad dS_0' = dx_0'\, dy_0' .$$

inally, adding up the contributions from the source in he final integral gives

$$x_i, y_i) = \iiiint_{-\infty}^{\infty} \left[\iint_{-\infty}^{\infty} \hat{u}(x_0, y_0)\hat{u}^*(x_0', y_0')\, d\sigma \right]$$
$$\cdot\hat{t}(x_i - x_0, y_i - y_0)\hat{t}^*(x_i - x_0', y_i - y_0')$$
$$\cdot\hat{o}(x_0, y_0)\hat{o}^*(x_0', y_0')\, dS_0\, dS_0' . \tag{3}$$

Now, the term inside the brackets is Hopkins' partial oherence factor $\gamma(x_0, y_0; x_0', y_0')$ which, for a plane source, s given by

$$(x_0, y_0; x_0', y_0') = \iint_{-\infty}^{\infty} \Gamma(\mu, \nu)e^{i[\mu(x_0 - x'_0) + \nu(y_0 - y'_0)]}\, d\mu\, d\nu \tag{4}$$

here $\Gamma(\mu, \nu)$ describes the intensity variation and eometry of the source, and is zero outside the source. can be defined in terms of the contrast of the Young's fringes in the (μ, ν)[2] plane formed by pinholes at (x_0, y_0) and (x_0', y_0') when illuminated by a source whose geometry and intensity variation is given by $\Gamma(\mu, \nu)$. Substitution of (4) into (3) results in the intensity distribution over the image plane in the compact form

$$i(x_i, y_i) = \iiiint_{-\infty}^{\infty} \gamma(x_0, y_0; x_0', y_0')$$
$$\cdot\hat{t}(x_i - x_0, y_i - y_0)\hat{t}^*(x_i - x_0', y_i - y_0')$$
$$\cdot\hat{o}(x_0, y_0)\hat{o}^*(x_0', y_0')\, dS_0\, dS_0' . \tag{5}$$

Though (5) appears as a rather formidable formulation of the final image intensity variation, the problem simplifies considerably when either of the two extremes of complete coherence and incoherence are considered.

Frequently it is convenient to carry out the analysis of a system, not in terms of the independent space variables, but rather in the corresponding spatial frequency domain. Hence, proper manipulation with Fourier integrals, similar to Hopkins' treatment of Fourier series, yields

$$i(x_i, y_i) = \iiiint_{-\infty}^{\infty} \tau(\mu, \nu; \mu', \nu')\hat{O}(\mu, \nu)\hat{O}^*(\mu', \nu') \cdot$$
$$\cdot e^{i[(\mu - \mu')x_i + (\nu - \nu')y_i]}\, d\mu\, d\nu\, d\mu'\, d\nu' \tag{6}$$

where

$$\tau(\mu, \nu; \mu', \nu') = \iint_{-\infty}^{\infty} \Gamma(s, t)$$
$$\cdot\hat{\tau}(s + \mu, t + \nu)\hat{\tau}^*(s + \mu', t + \nu')\, ds\, dt \tag{7a}$$

and

$$\Gamma(s, t) = \iint_{-\infty}^{\infty} \gamma(x_0, y_0)e^{-i(sx_0 + ty_0)}\, dx_0\, dy_0. \tag{7b}$$

With this generalized formulation in mind, we will now focus our attention on the problems of the Fourier analysis and synthesis of optical systems which are linear in intensity (incoherent) and linear in amplitude (coherent). Before distinguishing between the analysis and synthesis of incoherent systems, it seems appropriate at this point to discuss the physical significance of completely incoherent illumination with respect to (5), (6), and (7). To begin with, when dealing with a self-luminous object, we should expect only point-to-point coherence or incoherence since every point in object space radiates independently of every other point. Mathematically, such a situation can be described in terms of the well-known Dirac "delta-

[2] The relation between the cartesian coordinates (u, v) in Fig. 1 and the reduced coordinates (μ, ν) is

$$\mu = \frac{ku}{p}, \qquad \nu = \frac{kv}{p}$$

where $k = 2\pi/\lambda$ is the wave number of the radiation and p is the distance between the object and aperture plane in Fig. 1.

function." Thus describing the partial coherence factor by the delta function and making use of (7b), we can, conceptually at least, replace the self-luminous object by a transparent one illuminated by an "effective source," infinite in extent. In a similar fashion, for a transparent object illuminated by a large extended source, we can consider such an object self-luminous for all practical purposes, provided the source is large enough [8], [9]. Since this type of illumination is the one most frequently encountered, most of the effort thus far in optics has been directed toward the analysis of incoherent systems.

Incoherent Illumination

Analysis: In a broad sense, the analysis of a linear system can be defined as the determination of the Fourier frequency components that the system passes to make up the output. In the optical case, the sine wave response of the system is determined as a function of the spatial frequency from a knowledge of the geometry and complex transmission of the aperture; *e.g.*, coating, aberrations, etc.

To illustrate this, consider (5) where, for the incoherent case, the partial coherence factor $\gamma(x_0, y_0; x_0', y_0')$ is a "delta" function. This gives

$$i(x_i, y_i) = \iiiint_{-\infty}^{\infty} \delta(x_0 - x_0', y_0 - y_0') \cdot \hat{t}(x_i - x_0, y_i - y_0)\hat{t}^*(x_i - x_0', y_i - y_0') \cdot \hat{o}(x_0, y_0)\hat{o}^*(x_0', y_0')\, dx_0\, dy_0\, dx_0'\, dy_0'. \tag{8}$$

Integrating over $dx_0' dy_0'$ and making use of the sifting property of the "delta" function, we have

$$i(x_i, y_i) = \iint_{-\infty}^{\infty} |\, \hat{t}(x_i - x_0, y_i - y_0)\,|^2\, |\, \hat{o}(x_0, y_0)\,|^2\, dx_0\, dy_0 \tag{9}$$

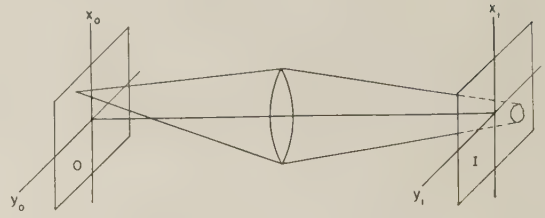
or

$$i(x_i, y_i) = \iint_{-\infty}^{\infty} t(x_i - x_0, y_i - y_0)o(x_0, y_0)\, dx_0\, dy_0 \tag{10}$$

where we see that the system is linear in intensity. (See Fig. 2.) The convenience of the Fourier approach becomes apparent when we perform a Fourier transformation of (10) to the spatial frequency domain and in the process replace the rather cumbersome convolution integral by a multiplication process. This gives

$$I(\mu, \nu) = \tau(\mu, \nu)O(\mu, \nu). \tag{11}$$

Thus we may consider the transfer or system function $\tau(\mu, \nu)$ operating on the spatial spectrum of the object intensity distribution to produce the spatial spectrum of the image intensity distribution. The term $\tau(\mu, \nu)$ has been assigned a variety of names in recent years, and, in fact, may be defined in a number of ways. For the sake of clarity and simplicity, we will define it in this paper as a measure of the reduction in contrast, and relative phase shift in passing from object to image space, of a sinusoidal



$$i(x_i, y_i) = \iint_{-\infty}^{\infty} t(x_i - x_0, y_i - y_0)o(x_0, y_0)\, dx_0\, dy_0$$

$$I(\mu, \nu) = \tau(\mu, \nu)O(\mu, \nu)$$

Fig. 2—A linear incoherent optical system.

intensity variation of increasing spatial frequency. It units along the abscissa will be in reciprocal length (lines/ mm), and it will be called the "transfer function."

A review of the literature [16], [18], [23-25] shows tha most of the attention thus far in recent years has bee focussed upon the evaluation of $\tau(\mu, \nu)$. The reason for thi is apparent when we consider the various Fourier relation that exist in systems that are diffraction limited. Fo example, one way of defining $\tau(\mu, \nu)$ mathematically is i terms of the Fourier transform of the point image intensit distribution; *i.e.*,

$$\tau(\mu, \nu) = \iint_{-\infty}^{\infty} t(x_i, y_i)e^{i(\mu x_i + \nu y_i)}\, dx_i\, dy_i \tag{12a}$$

or

$$\tau(\mu, \nu) = \iint_{-\infty}^{\infty} \hat{t}(x_i, y_i)\hat{t}^*(x_i, y_i)e^{i(\mu x_i + \nu y_i)}\, dx_i\, dy_i \tag{12b}$$

which, after application of a well-known theorem for th Fourier transform of a product, gives

$$\Gamma(\mu, \nu) = \iint_{-\infty}^{\infty} \hat{\tau}(\mu', \nu')\hat{\tau}^*(\mu' + \mu, \nu' + \nu)\, d\mu'\, d\nu' \tag{13}$$

where $\hat{\tau}(\mu', \nu')$ describes the complex transmission an geometry of the aperture and is zero outside the aperture For a uniform circular aperture with aberrations, we hav the following:

$$\hat{\tau}(\mu', \nu') = \begin{cases} e^{ik\Delta(\mu', \nu')} & \mu', \nu' \subset A \\ 0 & \mu', \nu' \not\subset A \end{cases} \tag{14}$$

where A denotes the area of the aperture and wher $\Delta(\mu', \nu')$ is the deviation of the actual wave front from reference sphere (the aberrations). Because of this apparen simplification, it should be possible from lens design dat to substitute the proper coefficients (*e.g.*, Seidel aberratio coefficients) into the expression for $\Delta(\mu', \nu')$ and perfor the convolution operation described by (13). In practice of course, because of the form of (14), the integral ca rarely be evaluated in closed form. Nevertheless, despit the practical difficulties associated with this approach th

reatment of an incoherent optical system as a two-imensional low-pass spatial frequency filter has strong onceptual appeal.

One advantage of such an approach, (13) and (14), is hat in theory at least, one can evaluate the transfer nction from a knowledge of the aberration coefficients ithout ever examining in detail the complicated physical ffraction pattern.

Synthesis and Limitations: In place of the passive role he assumes in analyzing linear systems, one might wish o take a more active part in altering the system's per-rmance by changing the image forming properties of he system and, hence, the transfer function. One method f doing this is to apply complex amplitude coatings over he aperture [10], [13], and [19] and hence alter the point nage distribution. An extreme in this direction would be o place a central obstruction in the aperture so that the ansparent area is annular [18], [25]. In fact, it might be ossible to optimize the frequency response within a iven prespecified spatial bandwidth of particular interest.[3]

Although several of these schemes have met with some uccess for special problems, the flexibility afforded to patial filtering operations through generalized synthesis s quite limited with incoherent systems. A little reflection ill show that this must be the case. Basically, we are ealing with the addition of nonnegative intensity varia-ons everywhere in the image plane which will always ive rise to background or constant levels, so that all ncoherent systems behave as low-pass spatial frequency lters.[4] Bearing this in mind, we are led to the obvious onclusion that general spatial filtering can be performed nly in systems in which destructive as well as con-tructive interference can occur. That is, we must be able o control amplitude and phase and, therefore, we are ed to a consideration of the image forming properties of oherent systems.

oherent Illumination

Analysis: It might be well to preface this section with he statement that the optical system to be discussed here ould be described by the microscopist as one employing point source Kohler illumination." In terms of the quations already derived for the more general case, it is ow necessary to impose the condition that, for a point ource, $\Gamma(\mu, \nu)$ is given by the Dirac delta function. Under his condition (5) becomes

$$i(x_i, y_i) = \int_{-\infty}^{\infty}\int \hat{t}(x_i - x_0, y_i - y_0)\hat{o}(x_0, y_0)\, dx_0\, dy_0$$

$$\cdot \int_{-\infty}^{\infty}\int \hat{t}^*(x_i - x_0', y_i - y_0')\hat{o}^*(x_0', y_0')\, dx_0'\, dy_0' \qquad (15a)$$

or

$$i(x_i, y_i) = \hat{i}(x_i, y_i)\hat{i}^*(x_i, y_i). \qquad (15b)$$

It is apparent that in terms of intensity, the system is nonlinear. Therefore, for convenience, the communication theory principles will be applied to a system linear in amplitude, tacitly assuming that (15b) will be needed to evaluate the resultant intensity variation.

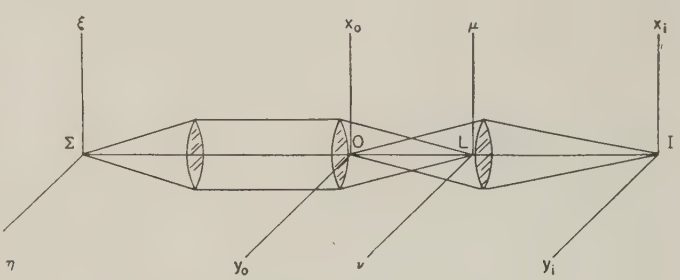
With this in mind, we can give the image amplitude distribution by the convolution integral of the point image amplitude distribution over the complex object amplitude distribution as follows:

$$\hat{i}(x_i, y_i) = \int_{-\infty}^{\infty}\int \hat{t}(x_i - x_0, y_i - y_0)\hat{o}(x_0, y_0)\, dx_0\, dy_0. \qquad (16)$$

Here again it is possible to replace the convolution integral by a multiplication process by transforming to the spatial frequency domain. This gives[5]

$$\hat{I}(\mu, \nu) = \hat{\tau}(\mu, \nu)\hat{O}(\mu, \nu). \qquad (17)$$

Again this equation may be interpreted as an operation on the object amplitude spectrum to give the image amplitude spectrum. Now, however, we see that the amplitude transfer function is identically given by $\hat{\tau}(\mu, \nu)$ which describes the geometry and complex transmission of the aperture in the (μ, ν) plane. Thus it becomes apparent that the Fourier components that make up the image can be controlled by inserting the proper mask in this plane. (See Fig. 3.)



$$\hat{i}(x_i, y_i) = \int_{-\infty}^{\infty}\int \hat{t}(x_i - x_0, y_i - y_0)\hat{o}(x_0, y_0)\, dx_0\, dy_0$$

$$\hat{I}(\mu, \nu) = \hat{\tau}(\mu, \nu)\hat{O}(\mu, \nu)$$

Fig. 3—A linear coherent optical system.

[3] For example, one possible criterion might be to minimize the ean square difference between the aberrated system and an ideal ne over a given spatial frequency range, *i.e.*,

$$\epsilon = \frac{1}{\omega_2 - \omega_1}\int_{\omega_1}^{\omega_2} (\tau_0 - \tau)^2\, d\omega$$

here τ_0 = transfer function in the ideal case (no aberrations) nd τ = actual transfer function. One might then determine the rrect balancing of aberration coefficients to make ϵ a minimum. he frequency range $\omega_1 \leq \omega \leq \omega_2$ would of course be determined r the application.

[4] This of course is no longer true for electro-optical systems in hich the constant term can easily be eliminated [11].

[5] This formulation can be arrived at by successively applying Huyghen's principle over the object and aperture planes as shown by Rhodes [21], [22] who in turn discusses the conditions under which the Fourier transform relations are applicable.

Before investigating these possibilities, it seems proper to verify the statements given above with a few simple examples. Consider first of all a clear unaberrated circular aperture. Then $\hat{\tau}(\mu, \nu)$ is the familiar cylindrical "top-hat" distribution, which "filter-wise" signifies an ideal low-pass, two-dimensional filter. There exist several basic limitations to perfect imagery in the system under discussion. The first is due to the wave character of radiation itself [6]. If the condition is imposed that the incident and emergent waves in object space must both satisfy the scalar wave equation, there results an inequality in terms of the wavelength of the light and the spacing between detail in the object. For detail closer together than the wavelength of the light, the coefficient of the space variable in the exponent becomes real and negative, instead of imaginary, which results in waves that die out within a few wavelengths of the object. This limitation exemplifies a well-known principle in communication theory which deals with the inability of a carrier to transmit information concerning details closer together than the wavelength of the carrier.

The second limitation imposed is due to the finite dimensions of the aperture. That is, while all diffracted angles between zero and ninety degrees are possible, only those Fraunhofer or Fourier orders will enter into the formation of the image that intercept the aperture in the (μ, ν) plane. Since this point of view is merely a generalized Abbé theory of image formation in the microscope, it is a simple matter to verify Abbé's claim, that for periodic targets the resolution limit of the system can be doubled by employing oblique illumination.

In order to do this, the point source must be moved off axis, which results in a tipped wave passing through the object plane. However, this is equivalent to having axial illumination with a linear phase plate over the object; hence the complex transmission of the object becomes

$$\hat{o}(x_0, y_0) = \hat{o}'(x_0, y_0)e^{ik(\alpha_0 x_0 + \beta_0 y_0)} \tag{18}$$

where α_0, β_0 are the direction cosines of the incident tipped wave. The Fraunhofer spectrum in the μ, ν plane is given by

$$\hat{O}(\mu, \nu) = \iint_{-\infty}^{\infty} \hat{o}'(x_0, y_0)e^{i[(\mu + k\alpha_0)x_0 + (\nu + k\beta_0)y_0]}\, dx_0\, dy_0, \tag{19}$$

and, if we define the terms $\mu_0 = k\alpha_0$ and $\nu_0 = k\beta_0$ (19) becomes

$$\hat{O}(\mu + \mu_0, \nu + \nu_0) = \iint_{-\infty}^{\infty} \hat{o}'(x_0, y_0)e^{i[(\mu + \mu_0)x_0 + (\nu + \nu_0)y_0]}\, dx_0\, dy_0. \tag{20}$$

That is, the spectrum of the object remains the same but is shifted to a new center which is, of course, the geometrical image of the displaced point source.

The resultant amplitude in the image plane is now given by

$$\hat{\imath}(x_i, y_i) = \iint_{-\infty}^{\infty} \hat{\tau}(\mu, \nu)\hat{O}(\mu + \mu_0, \nu + \nu_0)e^{-i(\mu x_i + \nu y_i)}\, d\mu\, d\nu. \tag{21}$$

Changing variables, we obtain

$$\hat{\imath}(x_i, y_i) = e^{i(\mu_0 x_i + \nu_0 y_i)} \iint_{-\infty}^{\infty} \hat{\tau}(\mu' - \mu_0, \nu' - \nu_0) \cdot \hat{O}(\mu', \nu')e^{-i(\mu' x_i + \nu' y_i)}\, d\mu'\, d\nu'. \tag{22}$$

Hence, the transfer function in amplitude is given by the complex transmission of the aperture centered about the shifted position as shown in Fig. 4(a).

To illustrate this point further, consider a crossed sinusoidal target with spatial frequencies ω_x, ω_y in the x and y directions respectively, such that

$$\hat{o}'(x_0, y_0) = e^{-i(\omega_x x_0 + \omega_y y_0)}. \tag{23}$$

Eq. (20) becomes

$$\hat{O}(\mu + \mu_0, \nu + \nu_0) = \iint_{-\infty}^{\infty} e^{i[(\mu + \mu_0 - \omega_x)x_0 + (\nu + \nu_0 - \omega_y)y_0]}\, dx_0\, dy_0 \tag{24}$$

which, except for a constant, is the integral representation of a delta function, so that

$$\hat{O}(\mu, \nu) = c^2\delta[\mu - (\omega_x - \mu_0), \nu - (\omega_y - \nu_0)] \tag{25}$$

where c is a constant. Substituting this expression into (21) and making use of the sifting property of the delta function, we obtain

$$\hat{\imath}(x_i, y_i) = c^2\hat{\tau}(\omega_x - \mu_0, \omega_y - \nu_0)e^{-i(\omega_x x_i + \omega_y y_i)}e^{i(\mu_0 x_i + \nu_0 y_i)} \tag{26}$$

which becomes, after normalizing,

$$\hat{\imath}(x_i, y_i) = \tau(\omega_x - \mu_0, \omega_y - \nu_0)\hat{o}'(x_i, y_i)e^{i(\mu_0 x_i + \nu_0 y_i)}. \tag{27}$$

Now noting that the last term merely describes the obliquity of the wave, we see that the object structure is reproduced in the image but that the amplitude is modulated by $\hat{\tau}(\omega_x - \mu_0, \omega_y - \nu_0)$. This implies that if the Fourier or Fraunhofer orders do not fall within the region of the shifted aperture function $\hat{\tau}(\mu, \nu)$, the periodicity will not be reproduced in the image. (See Fig. 4.) Now, since the spectrum is shifted by an amount corresponding to the direction cosines of the incident illumination, the final contrast and resolution limit are functions of the complex transmission of the system, the angular size of the aperture, and the direction of illumination. In terms of the sine wave resolution limit of the system, it is necessary that ω_x satisfy the following inequality in order for the periodicity to be detected in the image:

$$\omega_x \leq \mu_0 + \mu_1 \tag{28}$$

where μ/k represents the direction cosine of incident illumination and μ_1/k the maximum angular dimension of the aperture. Then,

$$\frac{2\pi}{P_x} \leq k(\sin\theta_0 + \sin\theta_1) \tag{29}$$

or, at the resolution limit,

$$P_x = \frac{\lambda}{(NA)_{\text{obj.}} + (NA)_{\text{coll.}}}. \tag{30}$$

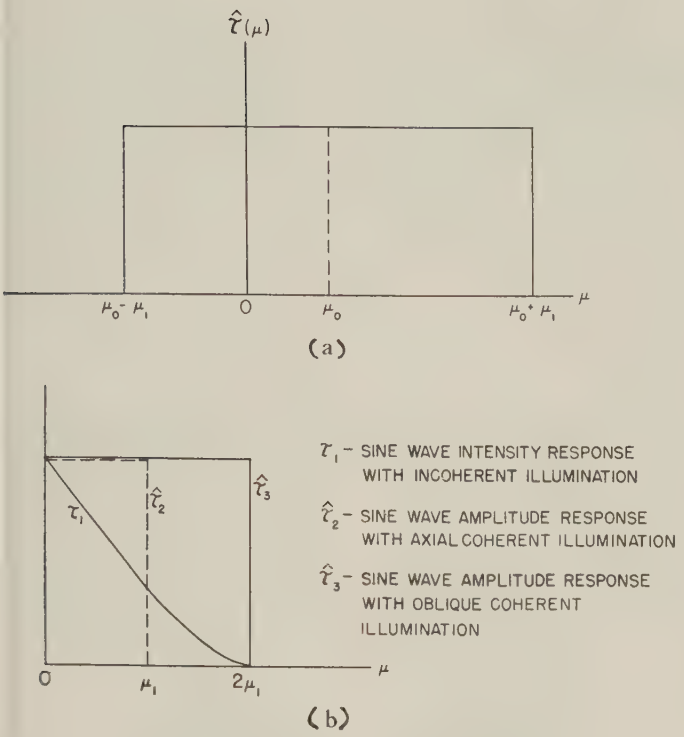


Fig. 4—Comparison of transfer functions.

For axial illumination

$$P_x = \frac{\lambda}{(NA)_{\text{obj.}}}. \tag{31}$$

For oblique illumination with the numerical aperture (NA) of the collimator equal to the numerical aperture of the objective,

$$P_x = \frac{\lambda}{2(NA)_{\text{obj.}}} \tag{32}$$

thus verifying Abbé's claim that the resolution limit for periodic structures can be doubled by employing the proper oblique illumination.

We have illustrated rather sketchily what might be termed the analysis of a linear coherent optical system. It now becomes clear that the synthesis of such a system is more promising than the synthesis of the previously discussed incoherent case. This is evident from an examination of (17) and Fig. 3, where it can be seen that the Fourier spectrum that constitutes the image can be controlled by inserting the proper mask in the (μ, ν) plane and hence controlling $\hat{\tau}(\mu, \nu)$. It is upon this aspect of the system that attention will be focussed in the remainder of the paper.

Before formulation of the communication theory approach to the problem of spatial filtering, it should be pointed out that this scheme of controlling the Fourier spectrum of the image is not new. The obvious illustration of this is the phase contrast microscope; however, the method first came to the attention of the author in an article by Maréchal and Croce [17] on increasing the contrast of photographs by a method known in communication theory as equalization [4]. Further investigation showed that Porter [20] in 1906 had performed a series of then remarkable spatial filtering experiments in verification of the Abbé theory of the microscope.

Synthesis: The purpose of the preceding section was to illustrate how a linear system can be analyzed by Fourier methods. It has been shown that the important contribution of such an analysis is that the rather cumbersome convolution process in the space domain can be replaced by the more practical multiplication process in the spatial frequency domain. It is now evident that the key to such an analysis is the transfer function $\hat{\tau}(\mu, \nu)$, which operates on the input spectrum to give the output spectrum, in this case the object and image respectively. In the synthesis processes to be discussed, it will be demanded that $\hat{\tau}(\mu, \nu)$ assume special characteristics in order to perform certain predescribed operations.

In the selection of $\hat{\tau}(\mu, \nu)$ in the remainder of this paper, the terminology and experience of electrical communication theory will be drawn on wherever applicable so that, to a large extent, the analogy between the electrical and optical systems will be emphasized.

A. Equalization: It is well-known in communication theory that if we have n linear systems in cascade, the final output spectrum is given by

$$F_{\text{out}}(\omega) = \hat{\tau}_1(\omega)\hat{\tau}_2(\omega) \cdots \hat{\tau}_n(\omega)F_{\text{in}}(\omega) \tag{33}$$

where $\hat{\tau}_k(\omega)$ is the transfer function for the k'th system. It should be noted in passing that Fourier methods really demonstrate their versatility in analyses of this kind since the n-fold multiplication process in the frequency domain is the counterpart of a series of n-fold convolution integrals in the space or time domain.

In the process of equalization we would like, in theory at least, to control $\hat{\tau}_n(\omega)$ in such a way that it becomes the reciprocal of the product of $\hat{\tau}_1(\omega)\hat{\tau}_2(\omega) \cdots \hat{\tau}_{n-1}(\omega)$; that is, we would like to design a $\hat{\tau}_n(\omega)$ such that

$$\hat{\tau}_n(\omega) = \frac{1}{\prod_{k=1}^{n-1} \hat{\tau}_k(\omega)}. \tag{34}$$

The reason for this is obvious when we examine (33). Of course in practice such control is not always possible. It was in this respect that Maréchal and Croce were able to increase the contrast of photographs by inserting in the (μ, ν) plane a mask that was dense at the center and gradually became transparent at the edge in such a way that the product of $\hat{\tau}(\omega)$ with the primary imaging transfer function gave a flat response. The result of inserting such a mask is shown in Fig. 5.

B. Edge Sharpening: The next step in altering the spatial spectrum of a photograph is the deletion of the constant term. This is similar to inserting a condenser in an electrical network and represents a high-pass filter. In terms of $\hat{\tau}(\mu, \nu)$ it merely means the insertion of a small opaque spot on axis in the (μ, ν) plane. The result of this operation

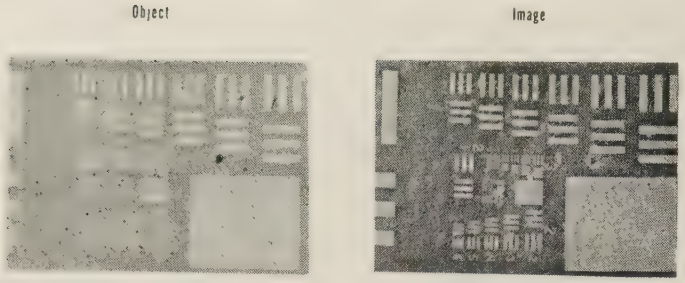


Fig. 5—Increase of contrast.

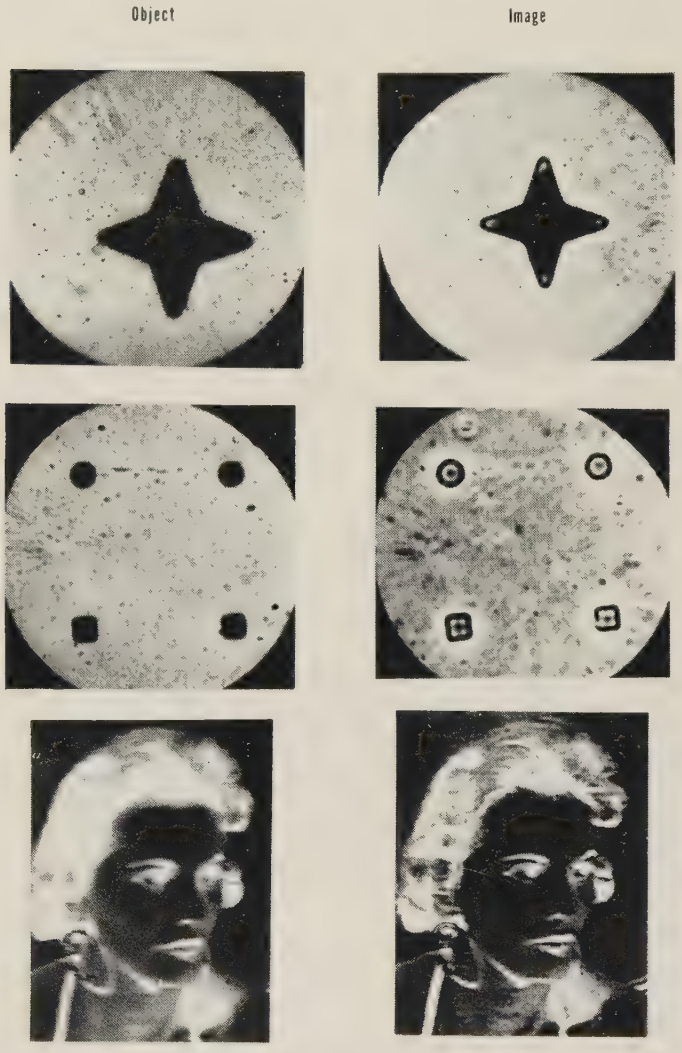


Fig. 6—Edge sharpening.

is shown in Fig. 6. Since, from an electrical standpoint the process is similar to differentiating the input, it would appear possible to add the edge-sharpened photograph to the original, optically or photographically. It should be noted that any operation such as edge sharpening or increasing signal to noise ratio for given geometrical shapes affects similar distributions anywhere in the field of view, since the Fraunhofer spectrum is formed on axis in the (μ, ν) plane. In addition to this favorable quality, the method has an additional property of performing filtering operations in two-dimensions at once, as opposed to electrical schemes.

C. Grain: In the special case where the distracting element in a photograph is the grain, these rapid fluctuations can be smoothed out as long as the size of the desired detail is not of the same order of magnitude. There are, of course, a number of other ways of doing this since all incoherent systems behave as low-pass spatial frequency filters. For example, the photograph could be blurred slightly or viewed out of focus. For the system under discussion, several photographs were chosen that appeared objectionably grainy to the eye. The aperture in the (μ, ν) plane was stopped down by a diaphragm until the visual image appeared noticeably smoother, and the image was then photographed. The result of rejecting these higher spatial frequency components due to grain is shown in Fig. 7. The effect is more pronounced when viewed with some magnification.

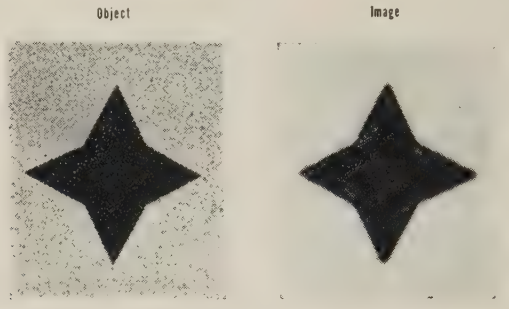


Fig. 7—Isolated detail with grain.

Detection and Recognition of Signals in the Presence of Noise: For convenience, in the remainder of this paper it will be assumed that the object function can be broken up into the sum of two separate functions, o_S and o_N, the signal and noise functions respectively. The reason for this is to maintain the electrical communication theory analog as far as possible. Bearing this analogy in mind, we will now subdivide the types of signals to be discussed into the three main categories of periodic, isolated (transient), and random detail.

A. Periodic Signals: It is a well-known principle in modern communication theory that an ideal method for recovering a periodic signal in noise is to cross correlate the signal plus noise distribution with a "Dirac comb;" *i.e.*, a series of evenly spaced sharp pulses at intervals corresponding to the fundamental period of the signal to be detected. This scheme has obvious applications in radar systems, for example. The optical equivalent of such a cross correlation procedure is to insert an opaque mask in the (μ, ν) plane with pinholes at positions corresponding to the spectral orders of the periodic portion of the target. The image forming lens recombines these spectral orders to produce the periodic disturbance in the image. (See Fig. 8.)

Loosely speaking, the opaque mask with the properly positioned pinholes can be represented as

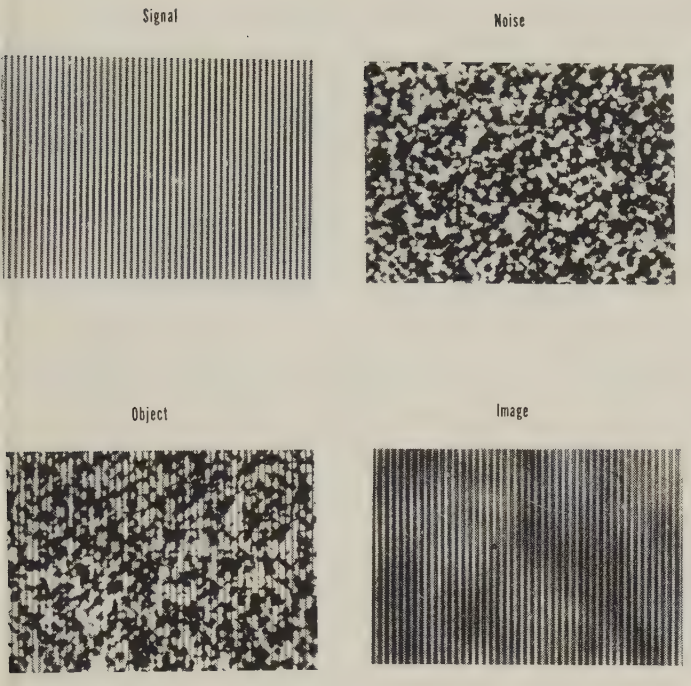


Fig. 8 –Periodic detail in the presence of "noise".

$$\hat{\tau}(\mu, \nu) = \sum_m \sum_n \delta(\mu - \omega_m, \nu - \omega_n) \tag{35}$$

where $\omega_m = m\omega_x$, and $\omega_n = n\omega_y$.

Multiplication of $\hat{\tau}(\mu, \nu)$ with the Fourier spectrum of signal and noise has the effect of passing the discrete spectrum of the periodic signal while sampling the continuous noise spectrum at discrete intervals. The result in the image plane is a reproduction of the periodic signal plus a small slowly varying term that oscillates about the average noise level. The calculation of the image distribution can actually be carried out for the case when there is little or no correlation between the noise and filter distributions [7]. Defining the total object distribution as a sum of signal and noise we have

$$\hat{o}(x_0, y_0) = \hat{o}_S(x_0, y_0) + \hat{o}_N(x_0, y_0) \tag{36}$$

where

$$\hat{o}_S(x_0, y_0) = \sum_m \sum_n c_{mn} e^{-i(\omega_m x_0 + \omega_n y_0)}.$$

The resultant image distribution is a convolution integral of the impulse response with the $\hat{o}(x_0, y_0)$ in (36). Because a Dirac comb is its own Fourier transform, the total integral is the sum of a convolution integral of a Dirac comb function with the periodic signal plus the convolution of the comb function with the noise. Because of the periodicity one may deal with average values of the integrals in terms of the cross correlation function. It can be shown quite readily that the cross correlation of a periodic signal with a Dirac comb function of the same period is, except for a constant, a reproduction of the periodic signal. By contrast, if the noise and comb function are uncorrelated, the result is a constant related to the average noise level. With this in mind, the image distribution becomes

$$\hat{\imath}(x_i, y_i) = K\hat{o}_S(x_i, y_i) + \epsilon. \tag{37}$$

B. Isolated Signals: The solution to this problem is not new to the communication engineer [7]; for this reason only an outline of the method will be presented here. The photointerpretive equivalent of maximizing the peak signal to rms noise ratio would be maximizing the probability of detecting an isolated piece of detail in unwanted background. The more selective recognition problem would seem to involve a sharpening or outlining operation. For the peak axial signal we may write

$$\text{peak axial signal} \sim \int_{-\infty}^{\infty}\int \hat{\tau}(\mu, \nu)\hat{o}_S(\mu, \nu)\, d\mu\, d\nu \tag{38}$$

and for the rms noise

$$\text{rms noise} \sim \left[\int_{-\infty}^{\infty}\int |\, \hat{\tau}(\mu, \nu)\,|^2\, d\mu\, d\nu\right]^{1/2} \tag{39}$$

so that the ratio becomes

$$\frac{\text{peak axial signal}}{\text{rms noise}} \sim \frac{\displaystyle\int_{-\infty}^{\infty}\int \hat{\tau}(\mu, \nu)\hat{o}_S(\mu, \nu)\, d\mu\, d\nu}{\left[\displaystyle\int_{-\infty}^{\infty}\int |\, \hat{\tau}(\mu, \nu)\,|^2\, d\mu\, d\nu\right]^{1/2}} \tag{40}$$

where again the subscripts S and N denote signal and noise respectively. Several independent investigators have shown that this ratio is a maximum when

$$\hat{\tau}(\mu, \nu) = \hat{o}_S^*(\mu, \nu) \tag{41}$$

that is, when the filter function is the complex conjugate of the desired signal spectrum. With such a filter, it can be readily shown that the image distribution is merely the autocorrelation function of the signal alone when signal and noise are uncorrelated.

An approximation to this filter can be achieved in the optical case by photographing the spectrum of the signal alone. The negative transparency can then be used as a mask when the signal and noise combination are viewed. (See Fig. 9). Note, however, that such a filter does not necessarily outline the signal as does the edge sharpening operation, but rather maximizes (40) and hence enhances the probability of detecting the signal. Finally, as Goldman [7] points out, for a given noise level, the ratio in (40) depends only upon the signal energy in the object. This would therefore appear to be a starting point for determining thresholds in terms of input (object) signal to noise ratio.

C. Random Signals: A detailed treatment of this problem is beyond the scope of this paper and present considerations. Nevertheless, for the sake of completeness, it proves of interest to examine how far we can go in ex-

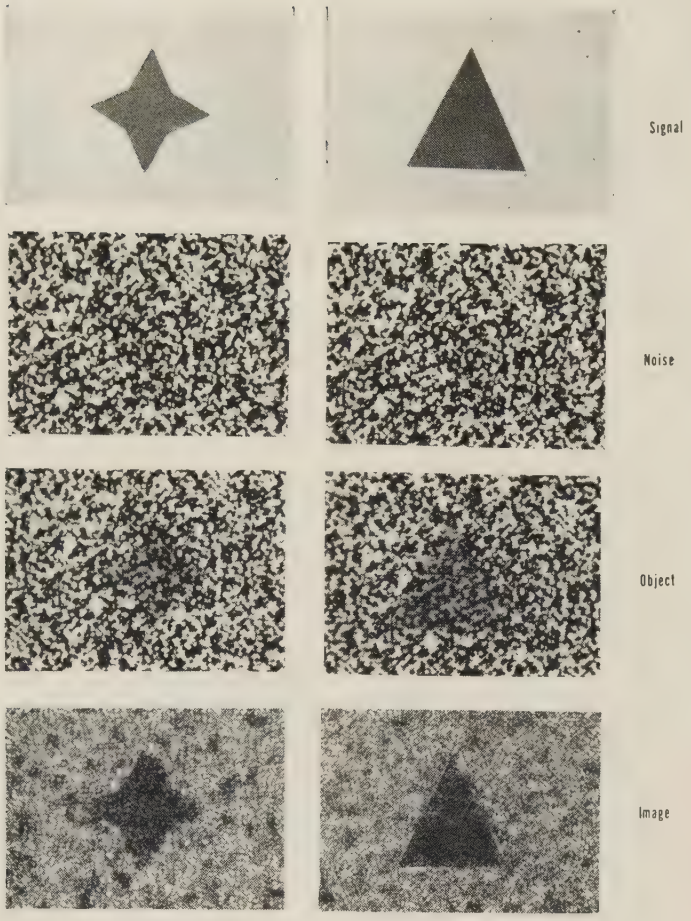


Fig. 9—Isolated signal in the presence of "noise".

tending the one-dimensional voltage vs time electrical considerations to the two-dimensional optical case. In such an extension, a suprising simplification is at once obvious in the optical problem. As pointed out by Cheatham and Kohlenberg [2], the fundamental difference between optical and electrical filters lies in the fact that once a filter function has been determined in the electrical case, one must then impose the condition that its response vanish for $t < 0$. This restriction is basic in that no amount of equipment or ingenuity can circumvent it. Such is not the case in the optical system. The impulse or point source response in optical systems lies on both sides of the space axes and in many cases possesses symmetry. From this point of view the design of an optimum filter presents some simplification in the optical case.

Conclusion

As pointed out recently by Linfoot and Fellgett [12], an optical system can be assessed from two distinct aspects. First of all, in terms of the primary imaging operation, it may be demanded that the intensity distribution in object space be transformed into an intensity distribution in image space with as high a degree of fidelity as possible. Secondly, assuming that it is impossible to exercise control over the primary imaging system, it may be demanded of a secondary imaging system that it extract as much useful information as possible from the photograph at the expense of unwanted background detail.[6] It is upon this latter aspect that attention has been focussed in this paper. Because the problem is not far removed from the problem of detecting signals in the presence of noise in electrical systems, the terminology of electrical network theory has been adopted wherever feasible.

A cursory examination was made of the image-forming properties of an optical system in general; the specialized cases of the analysis and synthesis of incoherent systems were discussed; and the limitations for such systems were mentioned, so far as spatial filtering is concerned. From an investigation of the coherent case, it became apparent that a method for performing spatial filtering could be carried out. To test this hypothesis, several experiments were attempted, in which the determination of the proper filter could be predicted from the equivalent electrical network problem. In fact, for many of the specialized problems of detecting certain geometrical patterns in the presence of unwanted background, the determination of the proper mask (filter) to insert becomes almost self-evident. Actually, for search problems such as this, the amount of information needed is analogous to the electrical case; that is, one must know either the desired pulse shape approximately or the statistical character of the noise (the power spectrum). In brief, if overlapping signal and noise are to be separated at all, they will be separated in their spectra, and the advantage of the optical over the electrical system is that such a separation takes place automatically in the optical case through the phenomenon of diffraction.

Finally, the same caution that the electrical engineer must exercise is warranted here except possibly to a larger extent. The insertion of a filter can often times give rise to false information in the image and unless properly interpreted may, in fact, negate the purpose of the filtering operation. While this information in the electrical case is most often put on a recording device and then studied objectively, it may happen that the optical image is examined visually which creates, as any microscopist knows, a number of psycho-physical problems. Whether or not this represents a basic limitation to the ideas discussed here remains to be determined.

Acknowledgment

The author is indebted to W. Wong for the bulk of the experimental work mentioned in this report and in particular for the production of the figures and diagrams used throughout the paper.

[6] This second aspect may imply reconstruction methods such as increasing contrast or edge sharpening, or it may possibly imply separation of usable and unwanted background detail in the photograph through spatial filtering.

Bibliography

[1] Blanc-Lapierre, A., "Upon Some Analogies Between Optics and Information Theory." Symposium on Microwave Optics, McGill University, Montreal, Canada, June, 1953.

[2] Cheatham, T. P., Jr., and Kohlenberg, A., "Optical Filters—Their Equivalence to and Difference from Electrical Networks." 1954 IRE Convention Record, part 4, "Electronic Computers and Information Theory," pp. 6–12.

[3] Duffieux, P. M., "L'Integrale de Fourier et ses Applications a l'Optique." Besancon, Faculte des Sciences, 1946.

[4] Elias, P., Grey, D. S., and Robinson, Z., "Fourier Treatment of Optical Processes." *Journal of the Optical Society of America*, Vol. 42 (February, 1952), p. 127.

[5] Elias, P., "Optics and Communication Theory." *Journal of the Optical Society of America*, Vol. 43 (April, 1953), p. 229.

[6] Gabor, D., "Lectures on Communication Theory." Technical Report No. 238, Research Laboratory of Electronics, Massachusetts Institute of Technology, April, 1952.

[7] Goldman, S., *Information Theory*. New York, Prentice-Hall, 1953.

[8] Hopkins, H. H., "The Concept of Partial Coherence in Optics." *Proceedings of the Royal Society A*, Vol. 208 (1951), pp. 263–277.

[9] Hopkins, H. H., "On the Diffraction Theory of Optical Images." *Proceedings of the Royal Society A*, Vol. 217 (1953), pp. 408–432.

[10] Jacquinot, P., Boughton, P., and Dossier, B., *Contributions in La Theorie des Images Optiques*. Paris, Editions de la Revue d'Optique, 1949, p. 183.

[11] Kovasznay, S. G., and Joseph, H. M., "Image Processing." Proceedings of the IRE, Vol. 43 (May, 1955), pp. 560–570.

[12] Linfoot, E. H., and Fellgett, P. B., "On the Assessment of Optical Images." *Royal Society of London Philosophical Transactions*, Series A, Vol. 247, p. 931.

[13] Luneberg, R. K., *Mathematical Theory of Optics*. Providence, R. I., Brown University, 1944.

[14] Macdonald, D. E., "Air Photography." *Journal of the Optical Society of America*, Vol. 43 (April, 1953), p. 290.

[15] Macdonald, D. E., "Quality Aspects of the Aerial Photographic System." National Bureau of Standards Circular 526, Proceedings of the N.B.S. Semicentennial Symposium on Optical Image Evaluations, 1951.

[16] Maréchal, A., "The Contrast of Optical Images and the Influence of Aberrations." National Bureau of Standards Circular 526, Proceedings of the N.B.S. Semicentennial Symposium on Optical Image Evaluations, 1951.

[17] Maréchal, A., and Croce, P., "Un Filtre de Frequences Spatiales pour l'Amelioration du Contraste des Images Optiques." *Comptes Rendus de Academie des Sciences*, Vol. 237 (September, 1953), pp. 607–609.

[18] O'Neill, E., "Transfer Function for an Annular Aperture." *Journal of the Optical Society of America*, Vol. 46 (1956), p. 285.

[19] Osterberg, H., and Wilkins, J. E., Jr., "The Resolving Power of a Coated Objective." *Journal of the Optical Society of America*, Vol. 39 (1949), p. 553.

[20] Porter, A. B., "On the Diffraction Theory of Microscopic Vision." *Philosophical Magazine*, Vol. 11 (1906), p. 154.

[21] Rhodes, J. E., Jr., "Analysis and Synthesis of Optical Images." *American Journal of Physics*, Vol. 21 (May, 1953), pp. 5, 337.

[22] Rhodes, J. E., Jr., "Microscope Imagery as Carrier Communication." *Journal of the Optical Society of America*, Vol. 43 (October, 1953), p. 848.

[23] Schade, O. H., "Electro-Optical Characteristics of Television Systems." *RCA Review*, Vol. 9, Nos. 1, 2, 3, and 4 (1948).

[24] Selwyn, E. W. H., "Theory of Resolving Power." National Bureau of Standards Circular 526, Proceedings of N.B.S. Semicentennial Symposium on Optical Image Evaluation, 1951.

[25] Steel, W. H., "Étude des Effets Combines des Aberrations et d'une Obturation Centrale de la Pupille Sur le Contraste des Images Optiques." *Revue d'Optique*, Vol. 32 (1953), pp. 4–26, 143–178, 269–306.

Effects of Signal Fluctuation on the Detection of Pulse Signals in Noise*

MISCHA SCHWARTZ†

Summary—**The Neyman-Pearson statistical theory on testing hypotheses has in previous work been applied to the problem of the detection of nonfluctuating constant-amplitude signals embedded in noise. This work is extended in this paper to the case of signal power fluctuating according to a prescribed probability distribution. The effect on system performance of possible correlation between successive signal pulses is taken into account.**

The introduction of signal fluctuation leads in general to some loss in system performance as compared to the case of nonfluctuating signals. This loss is most pronounced when there is complete correlation between successive signals, and is quite small when successive signals are independent of one another.

Introduction

THE PROBLEM of the detection of pulsed signals in the presence of noise has been treated rather extensively in the literature.[1-5] The Neyman-Pearson theory of testing statistical hypotheses[6,7] has been of particular value in much of this work; detailed analyses utilizing the Neyman-Pearson procedure have been made for the special case where the signal is assumed to be of constant amplitude.[1-3]

It is the aim of this paper to extend the analysis of signal detection to include the effect of signal amplitude fluctuations.

This is of particular significance in search radar systems where the signal received represents reflected energy from an airplane or other complex echoing target. Experimental investigations indicate that the signal amplitude in such cases may fluctuate through ranges of as much as 20-30 db as the aspect of the target changes with respect to the radar system.[8,9] If the turbulent motion of a plane is rapid enough compared to the radar system pulse repetition rate, wide fluctuations in reflected signal energy may take place from pulse to pulse.

The application of the Neyman-Pearson theory to constant-amplitude signal detection leads to a very simple threshold detection scheme for determining the presence of signal.[1-3] It can be shown quite simply that the same detection system results when the Neyman-Pearson procedure is applied to the case of rapidly-fluctuating signals.

This threshold system may be described as follows: A group of successive narrow-band IF pulses are detected by an envelope detector of the square-law type[10] and then summed by an appropriate integrating device. The resultant sum is required to exceed a fixed voltage threshold level to be labeled a signal. (Such a signal detection system was described in the literature by several authors before recognition of the procedure as a statistically optimum one in the Neyman-Pearson sense.[11-13]

The analysis of this system is carried out in this paper for three separate cases of signal fluctuation:

1) *Signals completely uncorrelated*: Here fluctuations in received signal power are so great from pulse to pulse that successive received signals may be treated as independent random quantities.

2) *Signals completely correlated*: The signal power received can only be defined statistically (*i.e.*, in a probability sense), but, once received on the first return pulse, it will remain constant over all pulses in the group. This would correspond to the case in airplane detection, for example, in which the plane motion is slow compared to the time taken for the antenna to move a beamwidth in azimuth.

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[1] M. Schwartz, "A Statistical Approach to the Automatic Search Problem," Ph.D. dissertation, Harvard Univ., Cambridge, Mass.; 1951.

[2] H. V. Hance, "The Optimization and Analysis of Systems for the Detection of Pulsed Signals in Random Noise," Sc.D. dissertation, Mass. Inst. Tech., Cambridge, Mass.; 1951.

[3] D. L. Drukey, "Optimum Techniques for Detecting Pulse Signals in Noise," presented at IRE National Convention, New York, N.Y.; March 4, 1952.

[4] T. G. Slattery, "Detection of a sine wave in the presence of noise by the use of a nonlinear filter," Proc. IRE, vol. 40, pp. 1232–1236; October, 1952.

[5] Middleton, "Statistical theory of signal detection," Trans. IRE, vol. IT-3, pp. 26–52; March, 1954.

[6] J. Neyman and E. S. Pearson, "On the problem of the most efficient tests of statistical hypotheses," *Proc. Roy. Soc. A*, vol. 231, pp. 289–337; 1931.

[7] J. Neyman and E. S. Pearson, "Testing of statistical hypotheses in relation to probability a priori," *Proc. Camb. Philo. Soc.*, vol. 29, pp. 492–510; 1933.

[8] D. Kerr, Ed., "Propagation of Short Radio Waves," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N.Y., vol. 13, ch. 6; 1951.

[9] J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N.Y., vol. 24; 1950.

[10] The analysis of the threshold detection system for the case of the constant-amplitude signals has shown that the exact detector characteristic is relatively unimportant in the detection process. Square-law detection is much more readily treated and is thus used in this analysis.

[11] S. M. Kaplan and R. W. McFall, "The statistical properties of noise applied to radar range performance," Proc. IRE, vol. 39, pp. 56–60; January, 1951.

[12] M. Schwartz, "A performance measurement criterion for search radar systems," *Sperry Engrg. Rev.*, vol. 3, pp. 20–25; July, August, 1950.

[13] J. C. Slater, "Report on noise and reception of pulses," M.I.T. Rad. Lab. Ser., sec. V, Rep. 25; January, 1941.

3) *Signals partially correlated*: This intermediate case is analyzed here only for the simplest choice of two samples added. The effects of video filtering, nonideal adding circuits, and antenna beam-shape are all neglected in this analysis.[1]

Threshold Detection Analysis

The threshold system can be described statistically in terms of two defined probabilities (related directly to the two types of error in the Neyman-Pearson tests).

These are: P_{nb}—the probability of integrated noise alone (in the absence of signal) exceeding a voltage threshold or bias level, and being labeled signal. A specified tolerable value for P_{nb} fixes the threshold level; and P_{sb}—the probability that the integrated signal in the presence of noise will exceed the chosen threshold level.

The over-all system performance (envelope detector, integrator, and threshold circuit) is rated in terms of the signal-to-noise (at the input to the detector) needed to achieve a given P_{sb} with P_{nb} fixed. With $P_{sb} = 90$ per cent, for example, and $P_{nb} = 10^{-10}$ (*i.e.*, the probability that noise will exceed the level is at any instant 1 part in 10^{10}), a certain signal-to-noise ratio, S/N, is needed if the signal is to be correctly detected (on the average) nine times out of ten that it appears. The specific value of S/N will of course depend upon the number of pulses, K, integrated.

Specifically, if $p_n(V)$ is the probability distribution (or density function) of noise voltage at the output of the detector-integrator combination and $p_s(V)$ the distribution of signal plus noise,

$$P_{nb} = \int_{b_s}^{\infty} p_n(V)\, dV \tag{1}$$

and

$$P_{sb} = \int_{b_s}^{\infty} p_s(V)\, dV \tag{2}$$

with b_s the threshold level in volts.

The noise distribution, $p_n(V)$, is readily determined. Thus, the envelope distribution of the Gaussian noise assumed at the detector input is given by the familiar Rayleigh distribution,

$$p_n'(r) = \frac{re^{-r^2/2\psi_0}}{\psi_0} \tag{3}$$

with ψ_0 the mean-squared noise voltage at the detector input.

The distribution at the output of a square-law envelope detector can then be readily shown to be

$$p_n'(V) = \frac{e^{-V/2a_s\psi_0}}{2a_s\psi_0} \tag{4}$$

where the detector output (in volts) is of the form

$$V = a_s r^2, \tag{5}$$

r the envelope voltage, and a_s a constant.

The convolution of this function K-fold times (assuming the noise samples added are independent quantities) gives the desired distribution at the output of the integrator[1]

$$p_n(x) = \frac{x^{K-1}e^{-x}}{(K-1)!} \tag{6}$$

with

$$x \equiv \frac{V}{2a_s\psi_0}.$$

P_{nb} is thus given by

$$P_{nb} = \int_{q_n}^{\infty} \frac{x^{K-1}e^{-x}}{(K-1)!}\, dx \tag{1a}$$

with $q_n = \dfrac{b_s/a_s}{2\psi_0}$, a normalized threshold level.

This integral has been tabulated by K. Pearson as the "Incomplete Gamma-Function."[14]

The determination of $p_s(V)$, the signal plus noise distribution at the output of the integrator, depends upon both signal and noise characteristics, and, in particular, upon the statistical properties of the signal.

The experimental data available on this latter point indicate that the reflected energy from almost all large complex targets is given by a proabability distribution of the form[8,9]

$$W(s) = \frac{e^{-S/\bar{S}}}{\bar{S}} \tag{7}$$

with S the instantaneous reflected power and $\bar{S}$ the average reflected power. ($\bar{S}$ depends upon the particular target involved.) This is a fortuitous result, since the distribution is of the same form as (4). The signal *voltage* amplitude before detection can thus be assumed to have the same Gaussian distribution as the noise, although of different mean-squared value. This distribution has also been obtained theoretically as the probability distribution of echoes from raindrops, sea clutter, etc.[8], and will be accepted as the distribution to be applied in most of the following analysis.

Uncorrelated Signal Fluctuations

Where the signal plus noise samples may be assumed uncorrelated from pulse to pulse, the voltage at the input to the detector will be the sum of two normally distributed (Gaussian) quantities—signal plus noise—or itself a normally distributed function with zero mean value and a variance given by the sum of the two variances:

$$\sigma^2 = \psi_0 + \sigma_s^2 = \psi_0(1 + s^2). \tag{8}$$

[14] K. Pearson, "Tables of Incomplete Gamma-Function," HM Stationery Office, London, Eng.; 1922.

Here ψ_0 is the mean-squared noise voltage, σ_s^2 is the mean-squared signal voltage (proportional to $\bar{S}$), and s^2 is the mean signal-to-noise ratio (power) at the detector input.

The analysis of the integration of K signal voltages at the detector output is thus identical to the procedure for the integration of K noise pulses, and the probability of signal detection may be written

$$P_{sb} = \int_{q_s}^{\infty} \frac{x^{K-1}e^{-x}\,dx}{(K-1)!}, \tag{9}$$

$$q_s = \frac{b_s}{2a_s\sigma^2} = \frac{q_n}{1+s^2}. \tag{10}$$

Eqs. (1a) and (9) serve to relate the two defined probabilities, P_{sb} and P_{nb}, to the mean signal-to-noise ratio, s^2, and the number, K, of pulses added. Thus, with P_{nb} specified, the threshold level (normalized to system noise power) is determined from (1a). Eq. (9) then gives P_{sb} as a function of s^2 and K.

The results for this case of uncorrelated fluctuating signal are plotted, for $P_{nb} = 10^{-10}$ and square-law detector, in Fig. 1. They are there compared with the results of a constant-amplitude signal analysis[1], the mean signal-to-noise ratio in the fluctuating case corresponding to the constant S/N ratio in the equal-amplitude case.

These curves indicate that the performance of the system is not impaired very much by fluctuation of successive signal echoes so long as K is great enough. For should one or two pulses reflected back be of low power the chances are that successive pulses will make up for this and still enable the signal to be detected. It is only for K a small number (less than 5) that the effect of fluctuation has, on the average, a deteriorating effect on performance. As P_{sb} increases from 50 per cent to 90 per cent the relative loss in performance due to fluctuation (or the additional signal power needed to maintain performance the same) increases. Thus, for $K = 2$, the fluctuating signal return requires about 0.8 db additional signal power to maintain the average performance at $P_{sb} = 50$ per cent. For $P_{sb} = 90$ per cent, this additional power required at $K = 2$ is 4.5 db.

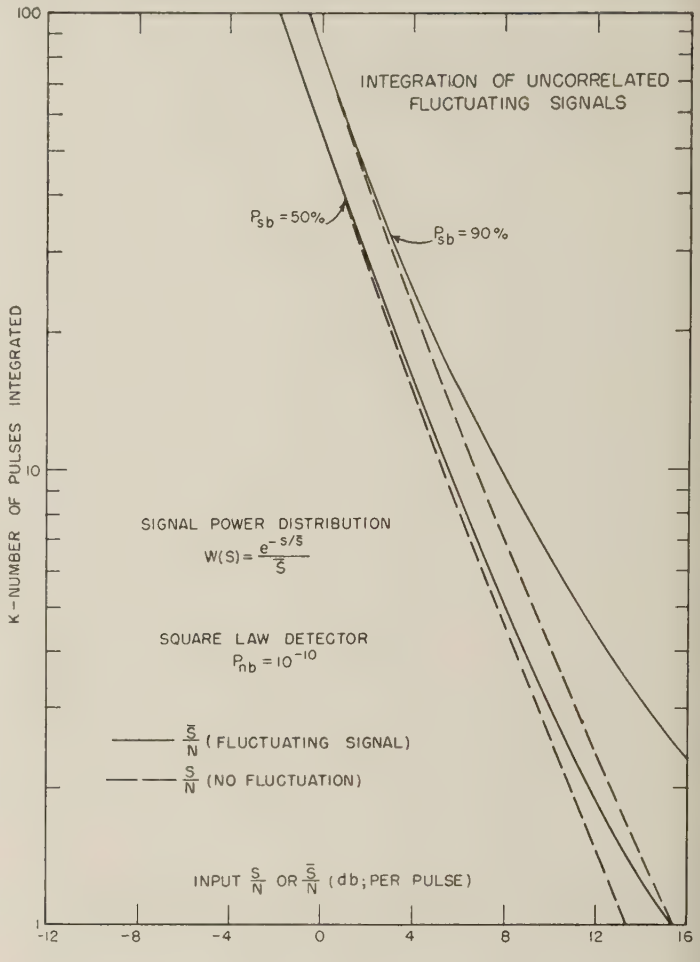

Fig. 1

Completely Correlated Signal Fluctuations

For the case of completely correlated fluctuating signal, the exact signal to be received is unknown, being given only by a probability distribution, but once it is received it remains fixed over all K pulses. This can then be solved using the results cited for constant-amplitude signals.[1]

With P_{nb} and K fixed, P_{sb} is a function of the power signal-to-noise ratio. This ratio in turn is directly proportional to the received signal power which is distributed according to a specified probability distribution, and which may take on (theoretically) all values from 0 to ∞. On the average, then, the probability of detecting a signal from a slowly-moving airplane will be given by

$$\overline{P_{sb}} = \int_0^{\infty} P_{sb}W(s)\,ds \tag{11}$$

where P_{sb} is the probability of detecting a given fixed signal S, and $W(S)dS$ is the probability of receiving this particular signal S (power).

Using the results of a constant signal analysis, P_{sb} may be plotted vs S (given P_{nb} and K) and the resultant curves approximated by simple functions. From these and a specified $W(S)$, then, the indicated integration of (11) may be performed and $\overline{P_{sb}}$ plotted vs $\bar{S}/N$. This procedure has been carried out for the aforementioned distribution

$$W(S) = \frac{e^{-S/\bar{S}}}{\bar{S}} \tag{7}$$

and also for another, arbitrarily chosen, distribution,

$$W(S) = \frac{Se^{-S^2/2S_0^2}}{S_0^2} \tag{12}$$

where $S_0 = \bar{S}\sqrt{2/\pi}$ and $\bar{S}$ is the mean value of S, as previously. Eq. (12) is seen to be the Rayleigh distribution for the distribution of noise at the output of a linear envelope detector.

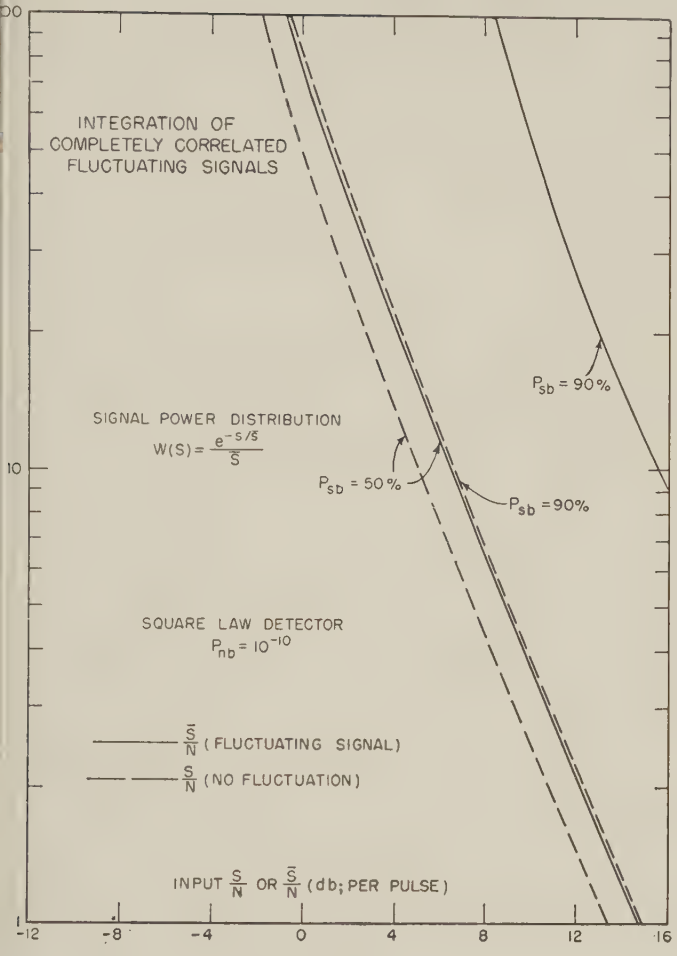

Fig. 2

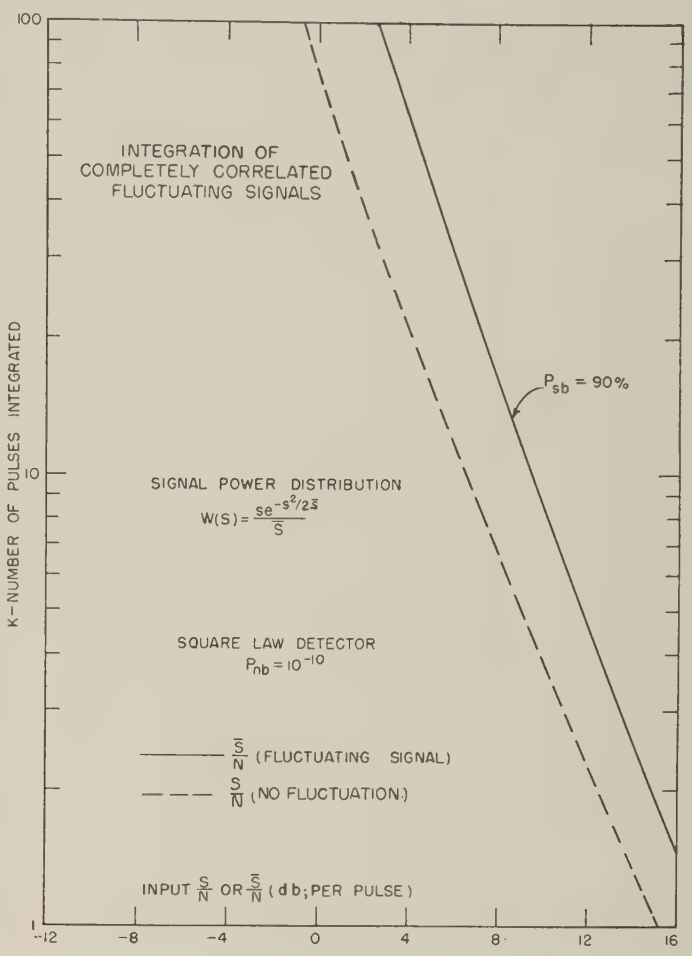

Fig. 3

The results obtained are shown plotted in Figs. 2 and 3 respectively, in which the constant signal (nonfluctuating) results are also included. These curves show that the effect of completely-correlated fluctuations is to produce a fairly constant loss in performance, the additional signal power required to recoup this performance (*i.e.*, give same P_{sb} and P_{nb}) being very nearly the same for all K. As expected also, the distribution of (7) produces a greater fluctuation loss than that which would result from a distribution given by (12): The distribution given by (12) has a much smaller deviation about its mean value and so approaches more nearly the δ-function distribution of a constant nonfluctuating signal.

The losses due to completely-correlated fluctuations are much worse than those in the uncorrelated case, as shown by comparing Figs. 1 and 2. This is due to the fact that once a given signal power has been received, complete correlation requires the successive signals to be of the same magnitude in power. A low echo return would thus render the detection of the signal very improbable, while for the uncorrelated case this low echo could conceivably be followed by a much higher one on the next pulse.

Partially Correlated Fluctuating Signals

Although the calculations for the two extreme cases of fluctuating signal just treated show the effect of signal fluctuation on system performance in terms of the Neyman-Pearson derived probabilities, P_{sb} and P_{nb}, it is apparent that in general the actual correlation for practical systems will lie somewhere between these two extremes. Typical echo data and correlation functions derived therefrom are shown in the literature, the correlation approaching zero rapidly as the time interval between the successive voltage is increased.[15]

This analysis will treat only the simplest case of two signal (or noise) samples added with the sum required to exceed a specified threshold level. Using these results and those previously obtained for the completely-correlated and uncorrelated cases, conclusions may be drawn as to the effect of correlation between successive signal pulses on signal detection for K greater than two.

In this analysis the noise is again assumed uncorrelated, as are signal and noise. The singal voltage at the

[15] H. M. James, N. B. Nichols, and R. S. Phillips, "Theory of Servomechanisms," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N.Y., vol. 25, ch. 6; 1947.

input to the detector is assumed to be Gaussianly-distributed with normalized correlation function $-1 \leq \rho \leq 1$. The correlation function, ρ, is defined as

$$\rho = \frac{\overline{(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)}}{\sqrt{\sigma_1^2 \sigma_2^2}}, \tag{13}$$

where x_1 and x_2 are the amplitudes of the two successive samples, assumed normally distributed, $\bar{x}_1$ and $\bar{x}_2$ the mean values (here zero), and σ_1^2, σ_2^2 the variances of x_1 and x_2 (mean-squared signal voltage).

The correlation functions plotted in the literature represent correlation in signal power, or mean-squared voltage, of successive samples. It can be shown[1], however, that ρ_s, the power correlation, is given simply by

$$\rho_s = \rho^2. \tag{14}$$

Passing each pulse through a square-law envelope detector and adding two successive pulses with correlation ρ, the probability distribution of the sum of these two samples turns out to be given by[1]

$$p_s(V) = \frac{e^{-\alpha V/2\rho'}}{2\sigma^2\rho'} \sinh\left(\frac{\alpha V}{2}\right) \tag{15}$$

(the detector constant, a_s, has here been set equal to 1 for simplicity).
Here

$$\alpha = \frac{\rho'}{\sigma^2(1 - \rho'^2)}$$

$$\sigma^2 = \psi_0(1 + s^2),$$

as previously, and

$$\rho' = \frac{s^2\rho}{1 + s^2}.$$

ρ' can be interpreted as a modified correlation function arising from the sum of noise and signal before detection. The correlation of the two signal samples is thus decreased due to the (assumed) independent noise fluctuations. In particular, as the mean signal-to-noise ratio, s^2, becomes very small the over-all correlation function, ρ', approaches zero. On the other hand, for large signal-to-noise ratio, the signal voltages maintain control and the correlation is effectively ρ, that for the signal alone.

The Neyman-Pearson derived probability that the sum of these partially correlated variables will exceed a specified voltage threshold level, b_s, is then

$$P_{sb_2} = \int_{b_s}^{\infty} p_s(V)\, dV. \tag{16}$$

Using the exponential equivalent for the hyperbolic sine in (15), (16) may be readily integrated to give

$$P_{sb_2} = e^{-q_s/(1-\rho'^2)}\left\{\cosh\left[\frac{q_s\rho'}{1 - \rho'^2}\right] + \frac{1}{\rho'}\sinh\left[\frac{q_s\rho'}{1 - \rho'^2}\right]\right\}. \tag{16a}$$

Here

$$q_s = q_n/(1 + s^2),$$

$$q_n = \frac{b_s}{2a_s\psi_0},$$

as previously. As a check, with $\rho = 0$, $\rho' = 0$, and

$$P_{sb_2} = e^{-q_s}(1 + q_s), \tag{17}$$

in agreement with (9) specialized to $K = 2$.

As previously a particular choice of the Neyman-Pearson probability P_{nb_2} (the probability of the sum of two noise variates in the absence of signal exceeding the threshold level b_s) fixes q_n, and P_{sb_2} may thus be calculated as a function of signal-to-noise ratio for the specified value of P_{nb_2}.

This has been done, using (16a), and the results are shown plotted in Fig. 4 ($P_{nb_2} = 10^{-10}$) and Fig. 5 ($P_{nb_2} = 10^{-5}$) for values of the signal voltage correlation function ρ, from 0 to 1 (complete correlation). As stated previously $\rho^2 = \rho_s$, the correlation function of the reflected signal power.

These results are as would be logically expected from the physics of the problem: For an increasing mean signal-to-noise ratio (per pulse) the completely correlated case becomes progressively worse (in terms of the system performance standards) than the uncorrelated case. The results for partial correlation fall between these two extremes. This is so because if by chance the signal power on the first returned pulse should be much lower than the mean signal power, the signal power on the next pulse will be exactly the same, in the case of $\rho = 1$. If the signal can change from pulse-to-pulse, however (partial correlation, $\rho < 1$) then there is a chance the power returned on the succeeding pulse will be greater, increasing the signal detectability (P_{sb_2}).

But for small mean signal-to-noise ratio the probability distribution is much narrower about the mean value, so that for uncorrelated signals the chance of the second signal pulse being much higher than the first is not as great. Thus the system performances for the correlated and uncorrelated cases approach one another. (*i.e.*, for a "narrow" probability distribution the introduction of correlation does not change matters very much, on the average.) With the mean S/N small enough (for $P_{nb_2} = 10^{-10}$, $S/N < 10$ db), the performance under correlated conditions becomes in fact slightly better than with uncorrelated signals.

It is also noteworthy to remark that for $P_{sb_2} < 40$ percent (roughly—the actual value depends upon P_{nb_2}) the fluctuating signal cases become "better" than the constant signal case. This is due to the broader probability distribution of the signal plus noise under fluctuating conditions. For small mean S/N, higher voltages have a greater chance of occurring in the fluctuating case than in the constant signal case.

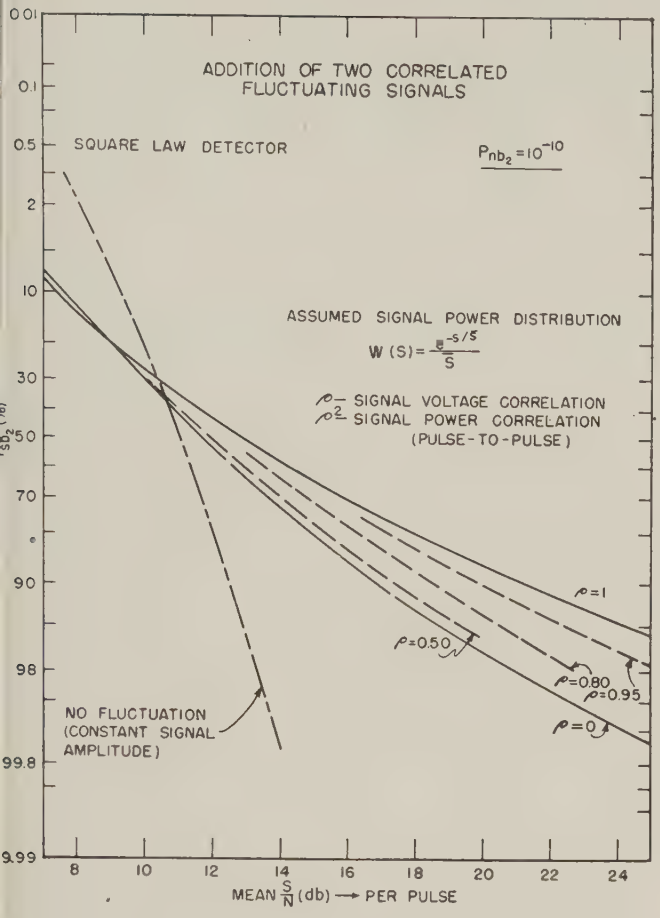

Fig. 4

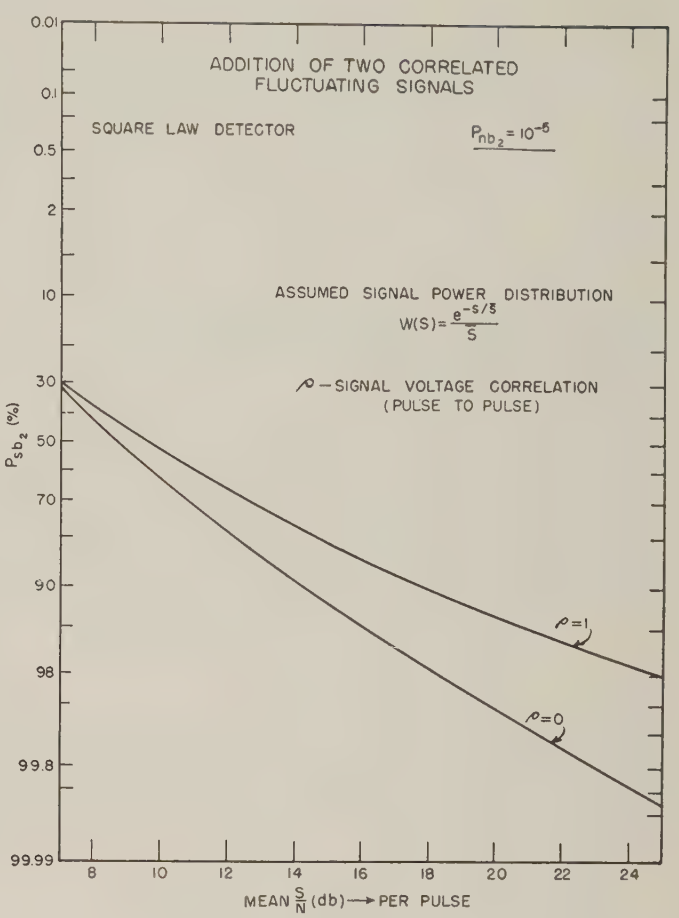

Fig. 5

These results for partial correlation of two successive gnals may also be profitably compared and checked ith the results determined previously and independently or the two extreme cases of $\rho = 0$ and $\rho = 1$ (Figs. 1 and). Thus, the results of Figs. 1 and 2 for the particular noice of $K = 2$ agree fairly well with those of Fig. 4 same signal-to-noise ratio for specified P_{sb_2}). This check particularly satisfying for the case of completely-orrelated signals ($\rho = 1$) since this case was previously nalyzed using a different approach than that followed ere [see (11)] and the results were obtained by rough pproximation procedures.

The method of analysis used here could presumably be xtended to the general case of partially correlated uctuating K pulses added together after detection, but ie mathematics involved would be extremely laborious. However, since the results for the partially correlated case fall between the two extremes of $\rho = 0$ and $\rho = 1$, for which the results have been plotted in Figs. 1, 2, and 3, these latter curves may be considered sufficient as bounds on the actual effect of fluctuation in practical systems.

A comparison of the two sets of curves, Figs. 4 and 5, indicates that the change in P_{nb_2} (or threshold level) over the range 10^{-5} to 10^{-10} (at least) does not affect the conclusions discussed above as to differences between the correlated and uncorrelated cases. This is true too for the previous results obtained for the two extreme cases for all K. The effect of increasing correlation (requiring progressively more power to give the same P_{sb} and P_{nb}, for $P_{sb} > 40$ per cent) is thus very nearly independent of P_{nb}.

Solution of an Integral Equation Occurring in the Theories of Prediction and Detection

K. S. MILLER† AND L. A. ZADEH‡

Summary—In many of the theories of prediction and detection developed during the past decade, one encounters linear integral equations which can be subsumed under the general form $\int_a^b R(t, \tau)\, x(\tau)\, d\tau = f(t)$, $a \leqq t \leqq b$. This equation includes as special cases the Wiener-Hopf equation and the modified Wiener-Hopf equation $\int_0^T R(|\, t - \tau\, |)\, x(\tau)\, d\tau = f(t)$, $0 \leqq t \leqq T$.

The type of kernel considered in this note occurs when the noise can be regarded as the result of operating on white noise with a succession of not necessarily time-invariant linear differential and inverse-differential operators. For this type of noise, which is essentially a generalization of the stationary noise with a rational spectral density function, it is shown that the solution of the integral equation can be expressed in terms of solution of a certain linear differential equation with variable coefficients.

Introduction

IN MANY of the theories of prediction and detection developed during the past decade, one encounters linear integral equations which can be subsumed under the general form

$$y(t) = \int_{a-}^{b+} R(t, \tau)x(\tau)\, d\tau, \qquad a \leqq t \leqq b, \tag{1}$$

where $R(t, \tau)$ denotes the auto-correlation function of a random, not necessarily stationary, process; $x(\tau)$ is the impulsive response of a filter (for fixed t); and a and b are constants, with the range of integration extending from $a-$ to $b+$ in order to contain any delta function terms which may be present at the end points a and b.

Typical special cases of (1) are: The Wiener-Hopf equation[1-3]

$$y(t) = \int_0^\infty R(t - \tau)x(\tau)\, d\tau, \qquad 0 \leqq t < \infty; \tag{2}$$

and the modified Wiener-Hopf equation

$$y(t) = \int_0^T R(t - \tau)x(\tau)\, d\tau, \qquad 0 \leqq t \leqq T, \tag{3}$$

which is encountered in prediction and smoothing,[4-7] detection,[8-13] maximum likelihood estimation,[14-16] and in the representation of stationary processes.[13-15,17] The modified Wiener-Hopf equation can be solved explicitly[4,8,11] when the spectral density function corresponding to $R(\tau)$ is rational in ω^2, and in a few other special cases including that of band-limited white noise.[15] The homogeneous form of (3) has been treated by De Sobrino,[9] Muller,[10] Slepian,[15] and, more recently and very comprehensively by Youla.[18]

Kernels of the more general form $R(t, \tau) \neq R(t - \tau)$ occur when the underlying random process is nonstationary.[19-22] In such cases it is not possible, in general, to

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[1] N. Wiener, "The Extrapolation, Interpolation, and Smoothing of Stationary Time Series," John Wiley & Sons, Inc., New York, N.Y.; 1949.

[2] Y. W. Lee, "Application of Statistical Methods to Communication Problems," Tech. Rep. No. 181, M.I.T. Res. Lab. of Elec.; 1950.

[3] H. P. Thielman, "On a class of singular integral equations occurring in physics," *Quart. Appl. Math.*, vol. 6, pp. 443–448; 1949.

[4] L. A. Zadeh and J. R. Ragazzini, "An extension of Wiener's theory of prediction," *J. Appl. Phys.*, vol. 21, pp. 645–655; July, 1950.

[5] J. H. Laning, Jr., "Prediction and Filtering in the Presence of Gaussian Interference," Tech. Rep. No. R-27, M.I.T. Instr. Lab. October, 1951.

[6] G. W. Preston, "The equivalence of optimum transducers and sufficient and most efficient statistics," *J. Appl. Phys.*, vol. 24, pp. 841–844; July, 1953.

[7] W. B. Davenport, Jr., R. A. Johnson, and D. Middleton, "Statistical errors in measurements on random time functions," *J. Appl. Phys.*, vol. 23, pp. 377–388; April, 1952.

[8] L. A. Zadeh and J. R. Ragazzini, "Optimum filters for the detection of signals in noise," PROC. IRE, vol. 40, pp. 1223–1231; October, 1952.

[9] R. DeSobrino, "Optimum Signal Detection with Incompletely Specified Signal and Noise," Ph.D. dissertation, Columbia Univ. New York, N.Y.; April, 1953.

[10] F. A. Muller, "Communication in the Presence of Additive Gaussian Noise," Tech. Rep. No. 244, M.I.T. Res. Lab. of Elec. May, 1953.

[11] Middleton, "The Statistical Theory of Detection," Tech. Rep. No. 35, M.I.T. Lincoln Lab.; November, 1953.

[12] E. Reich and P. Swerling, "The detection of a sine wave in Gaussian noise," *J. Appl. Phys.*, vol. 24; pp. 289–296; March, 1953.

[13] R. C. Davis, "The detectability of random signals in the presence of noise," TRANS. IRE, vol. IT-3, pp. 52–62; March, 1954.

[14] U. Grenander, "Stochastic processes and statistical inference," *Ark. f. Mat.*, vol. 1, pp. 195–277; October, 1950.

[15] D. Slepian, "Estimation of signal parameters in the presence of noise," TRANS. IRE, vol. IT-3, pp. 68–89; March, 1954.

[16] D. C. Youla, "The use of the method of maximum likelihood in estimating continuous modulated intelligence which has been corrupted by noise," TRANS. IRE, vol. IT-3, pp. 90–105; March, 1954.

[17] K. Karhunen, "Zur spektraltheorie stochastischer prozesse," *Ann. Acad. Sci. Fennicae*, vol. A1; 1946.

[18] D. C. Youla, "A Finite-Time Homogeneous Wiener-Hopf Integral Equation," Tech. Rep. No. 376, Microwave Res. Inst. Polytechnic Inst. of Brooklyn, Brooklyn, N.Y.; October, 1955.

[19] C. L. Dolph and M. A. Woodbury, "On the relation between Green's functions and covariances of certain stochastic processes and its application to unbiased linear prediction," *Trans. Amer. Math. Soc.*, vol. 72, pp. 519–550; May, 1952.

[20] R. C. Booton, Jr., "An optimization theory for time-varying linear systems with nonstationary statistical inputs," PROC. IRE, vol. 40, pp. 977–981; August, 1952.

[21] R. C. Davis, "On the theory of prediction of nonstationary stochastic processes," *J. Appl. Phys.*, vol. 23, pp. 1047–1053; September, 1952.

[22] A. H. Koschmann, "Time-Varying Filters for Non-Stationary Signals on a Finite Interval," Ph.D. dissertation, Purdue Univ. Lafayette, Ind.; 1954.

solve (1) explicitly. There are, however, certain special cases in which explicit solutions of (1) can be obtained. Dolph and Woodbury, in particular, have solved (1) for the case where $R(t, \tau)$ is the Green's function of a known differential operator.

A case of greater practical interest which is considered in this note is that of a nonstationary process which is generated by successive operations on white noise with differential and inverse-differential operators. Such a process is essentially the nonstationary analog of a stationary process which has a rational spectral density function. As is well-known, the latter type of process can always be generated by operating on white noise with the product of two time-invariant differential and inverse-differential operators.[23]

The approach described in the sequel constitutes in effect an extension of the spectrum shaping technique[4,24] to the nonstationary case. The use of this technique in the nonstationary case does not yield an explicit solution to (1) as it does in the stationary case.[4] Rather, it reduces the determination of $x(\tau)$ to the solution of a differential equation with variable coefficients. From the practical viewpoint, such a reduction is of advantage when one has access to an analog computer. For with such computers, it is generally much easier to solve a differential equation with variable coefficients than an integral equation of the form of (1).

Notations and Assumptions

To begin with, we note that it is sufficient to solve (1) for the special case where the left-hand member is of the form $\delta(t - \xi)$, that is,

$$\delta(t - \xi) = \int_{a-}^{b+} R(t, \tau)x(\tau)\, d\tau, \qquad a \leqq t \leqq b. \tag{4}$$

If the solution for this case is denoted by $R^{-1}(\tau, \xi)$, then by superposition the solution of (1) may be written as

$$x(\tau) = \int_{a-}^{b+} R^{-1}(\tau, \xi)y(\xi)\, d\xi. \tag{5}$$

The assumption that $R(t, \tau)$ is the auto-correlation function of a process which is generated by acting on white noise with a succession of differential and inverse-differential operators, implies that $R(t, \tau)$ is the composition of a kernel $\Gamma(t, \lambda)$ with itself, that is,

$$R(t, \tau) = \int_{-\infty}^{\infty} \Gamma(t, \lambda)\Gamma(\tau, \lambda)\, d\lambda, \tag{6}$$

where $\Gamma(t, \lambda)$ is the result of operating on a delta function $\delta(t - \lambda)$ with a succession of differential and inverse-differential operators,

$$\Gamma(t, \lambda) = L^{-1}MN^{-1}P \cdots Y^{-1}Z\delta(t - \lambda), \tag{7}$$

where $L, M, \cdots, Z$ are specified linear differential operators with variable coefficients.

[23] It is tacitly assumed here that only the spectral distribution of the process is of interest.

[24] H. W. Bode and C. E. Shannon, "A simplified derivation of linear least square smoothing and prediction theory," Proc. IRE, vol. 38, pp. 417–425; April, 1950.

For simplicity we shall assume that there are only two operators. The same techniques may be utilized in the case of n operators. Thus

$$\Gamma(t, \lambda) = L^{-1}M\delta(t - \lambda). \tag{8}$$

This case is sufficiently general for most practical purposes and it includes as special cases those treated in the literature.[25]

The operators L and M are assumed to be of orders l and m respectively. The formal adjoint of L is denoted by L^* and its impulsive response is denoted by $L^{-1}(t, \xi)$. Thus

$$L_tL^{-1}(t, \xi) = \delta(t - \xi), \qquad L^{-1}(t, \xi) = 0 \quad \text{for} \quad t < \xi, \tag{9}$$

where the subscript t is used to identify the variable on which L operates. In terms of $L^{-1}(t, \xi)$, an operational expression such as $L^{-1}g(t)$ may be written in the explicit form

$$L^{-1}g(t) = \int_{-\infty}^{\infty} L^{-1}(t, \xi)g(\xi)\, d\xi. \tag{10}$$

Solution of the Equation

As a preliminary to the solution of (4) we shall establish the following relation:

$$L_tR(t, \tau) = M_tM_t^*L^{-1}(\tau, t) \tag{11}$$

where M^* is the Lagrange adjoint of M. To obtain (11) we make use of (10) to write (8) in the explicit form

$$\Gamma(t, \lambda) = \int_{-\infty}^{\infty} L^{-1}(t, \xi)[M_\xi\delta(\xi - \lambda)]\, d\xi. \tag{12}$$

Substitution of this expression in (6) yields

$$R(t, \tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L^{-1}(t, \xi)[M_\xi\delta(\xi - \lambda)]L^{-1}(\tau, \zeta) \cdot [M_\zeta\delta(\zeta - \lambda)]\, d\xi\, d\zeta\, d\lambda \tag{13}$$

and operating on both sides of (13) with L results in

$$L_tR(t, \tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [L_tL^{-1}(t, \xi)][M_\xi\delta(\xi - \lambda)]L^{-1}(\tau, \zeta) \cdot [M_\zeta\delta(\zeta - \lambda)]\, d\xi\, d\zeta\, d\lambda. \tag{14}$$

Since

$$L_tL^{-1}(t, \xi) = \delta(t - \xi), \tag{15}$$

(14) reduces to

$$\begin{aligned} L_tR(t, \tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t - \xi)M_\xi\delta(\xi - \lambda)L^{-1}(\tau, \zeta) \cdot M_\zeta\delta(\zeta - \lambda)\, d\xi\, d\zeta\, d\lambda \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M_t\delta(t - \lambda)L^{-1}(\tau, \zeta) \cdot M_\zeta\delta(\zeta - \lambda)\, d\zeta\, d\lambda. \end{aligned} \tag{16}$$

[25] The assumption that $\Gamma(t, \lambda) = LM^{-1}\,\delta(t - \lambda)$ leads to similar results.

Now it can readily be shown that, for any reasonably well-behaved function $F(\tau, \lambda)$,

$$\int_{-\infty}^{\infty} F(\tau, \zeta)[M_\zeta \delta(\zeta - \lambda)]\, d\zeta = M_\lambda^* F(\tau, \lambda) \tag{17}$$

and

$$\int_{-\infty}^{\infty} F(\tau, \lambda)[M_t \delta(t - \lambda)]\, d\lambda = M_t F(\tau, t). \tag{18}$$

Consequently (16) may be written as

$$L_t R(t, \tau) = \int_{-\infty}^{\infty} M_t \delta(t - \lambda) M_\lambda^* L^{-1}(\tau, \lambda)\, d\lambda \tag{19}$$

or equivalently,

$$L_t R(t, \tau) = M_t M_t^* L^{-1}(\tau, t), \tag{20}$$

which is the desired result.

Our objective is now to obtain an expression for the inverse kernel $R^{-1}(\tau, \xi)$ which is the solution of (4), that is,

$$\int_{a-}^{b+} R(t, \tau) R^{-1}(\tau, \xi)\, d\tau = \delta(t - \xi), \qquad a \leqq t \leqq b. \tag{21}$$

Operating on the left-hand member of (21) with L_t yields, by virtue of (20),

$$\begin{aligned} &\int_{a-}^{b+} L_t R(t, \tau) R^{-1}(\tau, \xi)\, d\tau \\ &\quad = \int_{a-}^{b+} M_t M_t^* L^{-1}(\tau, t) R^{-1}(\tau, \xi)\, d\tau \\ &\quad = M_t M_t^* \int_{a-}^{b+} L^{-1}(\tau, t) R^{-1}(\tau, \xi)\, d\tau \\ &\quad = M_t M_t^* L_t^{*-1} R^{-1}(t, \xi). \end{aligned} \tag{22}$$

For the right-hand member of (21) we get

$$L_t \delta(t - \xi) + \sum_\alpha A_\alpha \delta^{(\alpha)}(t - a) + \sum_\alpha B_\alpha \delta^{(\alpha)}(t - b) \tag{23}$$

where the linear combination of delta functions of various orders arises as a result of possible discontinuities in the right-hand member at the end points $t = a$ and $t = b$. The coefficients $A_\alpha(\xi)$ and $B_\alpha(\xi)$ as well as the range of α are undertermined at this stage.

On combining (22) and (23) we obtain

$$\begin{aligned} M_t M_t^* L_t^{*-1} R^{-1}(t, \xi) &= L_t \delta(t - \xi) \\ &\quad + \sum_\alpha A_\alpha \delta^{(\alpha)}(t - a) + \sum_\alpha B_\alpha \delta^{(\alpha)}(t - b) \end{aligned} \tag{24}$$

which on letting

$$\psi(t, \xi) = L_t^{*-1} R^{-1}(t, \xi) \tag{25}$$

and

$$\Delta(t, \xi) = \sum_\alpha A_\alpha \delta^{(\alpha)}(t - a) + \sum_\alpha B_\alpha \delta^{(\alpha)}(t - b) \tag{26}$$

may be written more compactly as

$$M_t M_t^* \psi(t, \xi) = L_t \delta(t - \xi) + \Delta(t, \xi). \tag{27}$$

Regarding this as a differential equation on $\psi(t, \xi)$ and solving for $\psi(t, \xi)$ we have, formally,

$$\begin{aligned} \psi(t, \xi) &= (M_t M_t^*)^{-1}[L_t \delta(t - \xi) + \Delta(t, \xi)] \\ &\quad + \sum_\beta C_\beta \phi_\beta(t) \end{aligned} \tag{28}$$

where the ϕ_β are linearly independent solutions of $MM^*\phi = 0$ and the $C_\beta(\xi)$ are constants with respect to t. Thus, operating on $\psi(t, \xi)$ with L^* yields

$$\begin{aligned} R^{-1}(t, \xi) &= L_t^* M_t^{*-1} M_t^{-1} L_t \delta(t - \xi) \\ &\quad + L_t^* M_t^{*-1} M_t^{-1} \Delta(t, \xi) + \sum_\beta C_\beta L^* \phi_\beta(t). \end{aligned} \tag{29}$$

Now $\Delta(t, \xi)$ contains delta functions of order at most $l - 1$. Consequently the second term in (29) contains delta functions of order at most $2l - 2m - 1$. Thus the solution of (4) is given by

$$\begin{aligned} R^{-1}(t, \xi) &= L_t^* M_t^{*-1} M_t^{-1} L_t \delta(t - \xi) \\ &\quad + \sum_{\alpha=0}^{2l-2m-1} A_\alpha \delta^{(\alpha)}(t - a) \\ &\quad + \sum_{\alpha=0}^{2l-2m-1} B_\alpha \delta^{(\alpha)}(t - b) + \sum_{\beta=1}^{2m} C_\beta L^* \phi_\beta(t) \end{aligned} \tag{30}$$

and the undertermined coefficients A_α, B_α, C_β may be evaluated as in the stationary case[4] by substituting (30) in (4) and treating the resulting equation as an identity. In the case of (30), however, the determination of the $\phi_\beta(t)$ and the constants $A_\alpha(\xi)$, $B_\alpha(\xi)$, and $C_\beta(\xi)$ would in general require the use of an analog computer. We note that for the special case of the modified Wiener-Hopf equation, (30) reduces to the explicit expression

$$\begin{aligned} R^{-1}(t - \xi) &= \frac{L(-p^2)}{M(-p^2)} \delta(t - \xi) \\ &\quad + \sum_{\alpha=0}^{2l-2m-1} A_\alpha \delta^{(\alpha)}(t - a) + \sum_{\alpha=0}^{2l-2m-1} B_\alpha \delta^{(\alpha)}(t - b) \\ &\quad + \sum_{\beta=1}^{2m} C_\beta e^{-\epsilon_\beta t} \end{aligned} \tag{31}$$

where $p = d/dt$ is the Heaviside operator, $M(\omega^2)/L(\omega^2)$ is the spectral density function corresponding to the autocorrelation function $R(\tau)$, and the $e^{-\epsilon_\beta t}$ are the linearly independent solutions of the differential equation $M(-p^2)\, \phi = 0$. This solution of (3) is equivalent to that given by Zadeh and Ragazzini.[4]

An Example

To illustrate the main points of the methods described above it will suffice to consider a simple example which can be handled analytically. One such simple and yet nontrivial case is when the operators L and M are of the first order and have leading coefficients which are linearly increasing functions of time.

Specifically, let $L = tp$, $M = tp + k$ where $p = d/dt$ and k is a positive constant. In this case $L^* = -tp - 1$ and $M^* = -tp + k - 1$. The following expressions for the various quantities entering into (30) are readily derived:

$$M^{-1}(t, \xi) = 1(t - \xi)\frac{1}{\xi}\left(\frac{\xi}{t}\right)^k, \qquad M^{*-1}(t, \xi) = 1(\xi - t)\frac{1}{t}\left(\frac{t}{\xi}\right)^k \tag{32}$$

$$\Gamma(t, \lambda) = \delta(t - \lambda) + \frac{k}{\lambda}1(t - \lambda) \tag{33}$$

$$R(t, \tau) = \delta(t - \tau) + 1(t - \tau)\frac{k - k^2}{\tau} + 1(\tau - t)\frac{k - k^2}{t} \tag{34}$$

$$\phi_1(t) = t^{-k}, \qquad \phi_2(t) = t^{k-1}. \tag{35}$$

In the above expressions $1(t)$ is the unit step function.

With a knowledge of L, L^*, $M^{-1}(t, \xi)$ and $M^{*-1}(t, \xi)$ it is a simple matter to calculate the leading term in (30). Thus

$$R^{-1}(t, \xi) = \delta(t - \xi) + 1(t - \xi)\frac{k^2 - k}{1 - 2k}\frac{1}{\xi}\left(\frac{\xi}{t}\right)^k + 1(\xi - t)\frac{k^2 - k}{1 - 2k}\frac{1}{t}\left(\frac{t}{\xi}\right)^k + C_1 t^{-k} + C_2 t^{k-1} \tag{36}$$

and the delta function terms are absent by virtue of $2l - 2m - 1$ being negative. On substituting this expression in (21) and performing the necessary integrations, one obtains two equations on C_1 and C_2. Thus C_1 and C_2 can be determined. They are

$$C_1 = \frac{k^2(k-1)}{2k-1}a^{k-1}\frac{k\left(\frac{\xi}{b}\right)^{k-1} + (k-1)\left(\frac{b}{\xi}\right)^k}{k^2\left(\frac{a}{b}\right)^{k-1} - (k-1)^2\left(\frac{b}{a}\right)^k}$$

$$C_2 = \frac{k^2(k-1)}{2k-1}\frac{1}{\xi}\frac{1}{b^{k-1}}\frac{k\left(\frac{a}{\xi}\right)^{k-1} + (k-1)\left(\frac{\xi}{a}\right)^k}{k^2\left(\frac{a}{b}\right)^{k-1} - (k-1)^2\left(\frac{b}{a}\right)^k}.$$

Substitution of these in (36) yields an explicit expression for the inverse kernel $R^{-1}(t, \xi)$.

In terms of this kernel the solution to (1) with an arbitrary left-hand member $y(t)$ is given by

$$x(t) = \int_{a-}^{b+} R^{-1}(t, \xi)y(\xi)\, d\xi.$$

Generalization of the Class of Nonrandom Inputs of the Zadeh-Ragazzini Prediction Model*

MARVIN BLUM†

Summary—**The prediction theory presented in this paper is an extension of the prediction theory of Zadeh and Ragazzini[1]. It differs from their theory in that the nonrandom component of the input signal in the Zadeh-Ragazzini model is restricted to a polynomial of known degree n. In the theory developed here, the nonrandom component of the input signal may be any arbitrary linear function of a subset of known analytic functions where the subset of functions are known *a priori* but the linear relationship need not be. As in the previous solution, the determination of the impulsive admittance of the optimum predictor reduces to the solution of a modified Wiener-Hopf integral equation.**

Theoretical Discussion

THE impulsive admittance can be determined implicitly for a certain class of stationary signal and noise. The modifications of the previous solution for the more general nonrandom input functions are given. The usefulness of the theory is demonstrated for an input consisting of the general nonrandom component plus white noise.

Zadeh and Ragazzini considered the following problem: Given an input to a linear filter composed of a stationary component $M(t)$ plus a polynomial $P(t)$ of known degree n and a stationary noise component $N(t)$, determine the linear filter which will produce an optimum prediction of some linear function of the signal component.

The optimization consists of selecting that linear filter whose output is closest to a desired output in the sense that the ensemble mean error of prediction is zero and the square error of prediction is a minimum.

This paper considers the same type of model of the input except that the nonrandom component is not restricted to a polynomial of degree n, but can be any linear function of a subset of $n + 1$ functions, analytic in the region $0 \leqq t \leqq T + \mid \alpha \mid$, where the $n + 1$ functions are known *a priori*, but the linear combination need not be known. The quantity T is the finite memory of the filter and α is the prediction time.

Thus, the nonrandom component is given by

$$P(t) = \sum_{k=0}^{n} a_k' P_k(t) \tag{1}$$

where the $P_k(t)$ are known but the vector a_k' need not be known. The impulsive admittance of the optimum linear filter in the least square sense is found for this more general input model.

The solution differs with respect to the previous solution in that the minimization is taken with respect to one point only. The point selected is noted by $t = T_0$ where $0 \leqq T_0 \leqq T$.

Since the method of solution for the generalized model is essentially parallel to the solution obtained by Zadeh and Ragazzini, only the results are presented here. For a detailed solution see their original paper.

Let the input to the filter be given by $e_1(t)$ where

$$e_1(t) = S(t) + N(t). \tag{2}$$

The function $S(t)$ is the signal and $N(t)$ is a stationary random disturbance. The signal has two components:

$$S(t) = P(t) + M(t), \tag{3}$$

where the nonrandom component of the signal is

$$P(t) = \sum_{k=0}^{n} a_k' P_k(t), \tag{4}$$

and $M(t)$ is a stationary random component.

Let $S^*(t)$ be the desired output and consider an ideal predictor whose impulsive admittance is given by $k(t)$. Then the relationship between $S^*(t)$ and $S(t)$ can be represented by

$$S^*(t) = \int_{-\infty}^{+\infty} k(\tau) S(t - \tau)\, d\tau. \tag{5}$$

If $K(p)$ is the system response of $k(\tau)$, then

$$S^*(t) = K(p) S(t). \tag{6}$$

Let $H(p)$ and $W(t)$ be the system response and impulsive admittance, respectively, of the actual predictor and let $e_2(t)$ be the output of the actual predictor. Then

$$e_2(t) = \int_0^{\infty} W(\tau) e_1(t - \tau)\, d\tau = \int_0^{T} W(\tau) e_1(t - \tau)\, d\tau, \tag{7}$$

since $W(\tau)$ is taken as equal to zero when $\tau > T$. The error of prediction is given by

$$\epsilon(t) = e_2(t) - S^*(t). \tag{8}$$

The optimum filter is obtained by finding the function $W(\tau)$ such that

$$\langle \epsilon \rangle_{av} = 0 \quad \text{or} \quad \langle e_2(t) \rangle_{av} \equiv \langle S^*(t) \rangle_{av} \tag{9}$$

where $\langle\ \rangle_{av}$ represents the ensemble average and $\langle \epsilon^2 \rangle_{av}$ is a minimum.

* This report was prepared to document the results of a theoretical investigation into the filter problem of nonrandom inputs. It is an extension of a partial solution made previously by L. A. Zadeh and J. R. Ragazzini.

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[1] L. A. Zadeh and J. R. Ragazzini, "An extension of Wiener's theory of prediction," *J. Appl. Phys.*, vol. 21, pp. 645–655; July, 1950.

It is assumed that

$$\langle M(t)\rangle_{av} = \langle N(t)\rangle_{av} = 0 \tag{10}$$

or all t so that combining (2), (3), (4), (5), and (7) gives

$$\langle e_2(t)\rangle_{av} = \int_0^T \sum_{k=0}^{n} a_k' P_k(t-\tau) W(\tau)\, d\tau \tag{11}$$

nd

$$\langle S^*(t)\rangle_{av} = \int_{-\infty}^{+\infty} k(\tau) \sum_{k=0}^{n} a_k' P_k(t-\tau)\, d\tau. \tag{12}$$

Eq. (9) can now be written as

$$\sum_{k=0}^{n} a_k' \left[\int_0^T P_k(t-\tau) W(\tau)\, d\tau - \int_{-\infty}^{+\infty} k(\tau) P_k(t-\tau)\, d\tau \right] = 0. \tag{13}$$

Since (13) must hold for any arbitrary set of constants a_k', it follows that when $t = T_0$ then

$$\int_0^T P_k(T_0-\tau) W(\tau)\, d\tau = \int_{-\infty}^{+\infty} k(\tau) P_k(T_0-\tau)\, d\tau \equiv Q_k(T_0), \tag{14}$$

where $k = 0, 1, 2, \cdots n$.

Note that (14) imposes a system of $n+1$ constraints on the impulsive admittance $W(\tau)$. For example, if

$$P(t) = a_3' t^3 + a_2' t^2$$

the solution would involve only two constraints because there are only two unknown constants a_3 and a_2. There are not four constraints because the input contains a cubic component of time.

The functions

$$P(t) = a_1'(t^3 + t^2)$$

and

$$P(t) = a_1'(t^3 + 2t)$$

will each lead to a system containing one constraint. The required filter will be different for each function. Thus the optimum filter design makes use of all the *a priori* knowledge about the input function to reduce the mean square error of prediction by reducing the number of constraint equations of the form of (14). This error reduction is accomplished at the expense of making the filter more specialized since the filter will be optimum only for the narrower class of input functions.

By combining (2), (3), (5), and (7), the prediction error can be written

$$= \int_0^T W(\tau)[M(t-\tau) + N(t-\tau)]\, d\tau - \int_{-\infty}^{+\infty} k(\tau) M(t-\tau)\, d\tau. \tag{15}$$

The mean square value of ϵ is given by[2]

$$\sigma^2 = \langle \epsilon^2 \rangle_{av} = \lim_{L\to\infty} 1/L \int_0^L \epsilon^2\, dt. \tag{16}$$

Define

$$\phi_M(\tau) = \lim_{L\to\infty} 1/L \int_0^L M(t) M(t-\tau)\, dt \tag{17}$$

and

$$\phi_N(\tau) = \lim_{L\to\infty} 1/L \int_0^L N(t) N(t-\tau)\, dt. \tag{18}$$

It will also be assumed that there is no correlation between signal and noise:

$$\lim_{L\to\infty} 1/L \int_0^L N(t) M(t-\tau)\, dt = 0. \tag{19}$$

Substituting (15), (17), (18), and (19) into (16) gives

$$\sigma^2 = \int_0^T \int_0^T W(\tau_1) W(\tau_2) [\phi_M(\tau_1-\tau_2) + \phi_N(\tau_1-\tau_2)]\, d\tau_1\, d\tau_2 - 2 \int_{-\infty}^{+\infty} \int_0^T W(\tau_1) k(\tau_2) \phi_M(\tau_1-\tau_2)\, d\tau_1\, d\tau_2 + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k(\tau_1) k(\tau_2) \phi_M(\tau_1-\tau_2)\, d\tau_1\, d\tau_2. \tag{20}$$

The solution for $W(\tau)$ requires that the σ^2 of (20) be minimized subject to the $n+1$ constraints of (14). This is a standard isoperimetric problem of the calculus of variations. One is led to minimize

$$I = \sigma^2 - 2 \sum_{k=0}^{n} Q_k \lambda_k \tag{21}$$

where the λ_k are Lagrangian multipliers. By substituting (14) and (20), (21) can be written

$$I = \int_0^T W(\tau_1)\, d\tau_1 \cdot \left\{ \int_0^T W(\tau_2)[\phi_M(\tau_1-\tau_2) + \phi_N(\tau_1-\tau_2)]\, d\tau_2 - 2 \int_{-\infty}^{+\infty} k(\tau_2) \phi_M(\tau_1-\tau_2)\, d\tau_2 - 2 \sum_{k=0}^{n} \lambda_k P_k(T_0-\tau_1) \right\}. \tag{22}$$

The last term of (20) has been omitted since it is independent of $W(\tau)$.

Setting the variation of $I = 0$, it is found that the minimum σ^2 is obtained when $W(\tau)$ satisfies the integral equation

$$\int_0^T W(\tau)[\phi_M(t-\tau) + \phi_N(t-\tau)]\, d\tau = \sum_{k=0}^{n} \lambda_k P_k(T_0-t) + \int_{-\infty}^{+\infty} k(\tau) \phi_M(t-\tau)\, d\tau, \tag{23}$$

where $0 \leqq t \leqq T$.[3]

[2] The assumed equivalence of the time average and ensemble implies that both $M(t)$ and $N(t)$ are ergodic as well as stationary processes.

[3] In Appendix II, a modification of (23) as derived by the referee is presented for the more general case when the minimization is with respect to the whole time interval instead of just the point $t = T_0$.

For a prediction of the Lth derivative of the input function at $t = T_0 - \alpha$,

$$k(\tau) = \delta^L(\tau - \alpha) \tag{24}$$

where $\delta^L(u)$ is the Lth derivative of the Dirac function $\delta(u)$.

Applying (24) to (14) yields

$$Q_k = \frac{d^L}{d\tau^L} P_k(T_0 - \tau)\Big|_{\tau=\alpha} \equiv P_k^{(L)}(T_0 - \alpha)$$

$$= \int_0^T W(\tau)P_k(T_0 - \tau)\, d\tau \tag{25}$$

as the set of constraint relationships for the prediction of the Lth derivative of the input message at time $t = T_0 - \alpha$.

In general, (23) cannot be solved explicitly in closed form. However, for a certain class of functions $[\phi_M(t) + \phi_N(t)]$, a solution for $W(\tau)$ is available. The power spectra associated with $\phi_M(\tau)$ and $\phi_N(\tau)$ are given by

$$S_M(\omega^2) = \int_{-\infty}^{+\infty} \phi_M(\tau)e^{-j\omega\tau}\, d\tau \tag{26}$$

and

$$S_n(\omega^2) = \int_{-\infty}^{+\infty} \phi_N(\tau)e^{-j\omega\tau}\, d\tau. \tag{27}$$

Let

$$S(\omega^2) = G(j\omega)G(-j\omega) = S_M(\omega^2) + S_N(\omega^2) \tag{28}$$

such that $G(j\omega)$ and $1/G(j\omega)$ are analytic in the right half of the $j\omega$ plane. Let $S(\omega^2)$ be further restricted by the relation

$$S(\omega^2) = \frac{A(\omega^2)}{B(\omega^2)}. \tag{29}$$

where $A(\omega^2)$ and $B(\omega^2)$ are polynomials in ω^2.

$$S(\omega^2) = |\, G(p)\, |^2 \tag{30}$$

where $p = j\omega$, and

$$G(p) = \frac{Q(p)}{R(p)} = \frac{a_0 + a_1p + \cdots + a_mp^m}{b_0 + b_1p + \cdots + b_lp^l} \tag{31}$$

Then it can be shown that:

$$W(t) = [u(t) - u(t - T)]\Big\{\sum_{k=0}^{n} \lambda_k' \frac{[R(p)]^2}{A(-p^2)} [P_k(T_0 - t)]$$

$$+ C_0 + \frac{1}{2\pi} R(p) \int_{-\infty}^{+\infty} \frac{S_M(\omega^2)K(j\omega)}{A(\omega^2)} k(j\omega)R(-j\omega)e^{j\omega t} \tag{32}$$

$$+ \sum_{j=1}^{2m} B_j e^{\alpha_j t}\Big\} + \sum_{j=1}^{l-m} C_j \delta^{j-1}(t) + \sum_{j=1}^{l-m} D_j \delta^{j-1}(T - t).$$

$C_0u(t)$ arises from the operation of $R(p)$ on the discontinuities of $W(t)$ at the origin. In (33) the constant is included in the term $A_0u(t)$. In the more general form, using (34), the solution may not contain a constant and must be stated explicitly. If (34) contains a constant, the C_0 may be merged with it to form only one unknown constant.

For one class of functions the solutions to (34) are particularly simple. This class of functions G is defined by the properties: G is a linear vector space with basis $P_k(t)$ $K = 0, 1, 2, \cdots n$; G maps onto itself by any linear differential operator

$$D = \sum_{j=0}^{m} b_j \frac{d^j}{dt^j}$$

Let $P(t) = \sum_{k=0}^{n} a_kP_k(t)$ where the a_k are any set of complex numbers not all equal to zero; then $P(t)\epsilon G$ provided[4] $DP(t)\epsilon G$ i.e.

$$\sum_{j=0}^{m} \sum_{k=0}^{n} b_ja_k \frac{d^j}{dt^j} P_k(t) = \sum_{k=0}^{n} a_k'P_k(t)$$

where any two of the sets b_j, a_k, or a_k' can be chosen arbitrarily.

It is shown in the appendix that

1) The necessary and sufficient condition that $P(t)\epsilon G$ is that $d/dt\ P_k(t)\epsilon G$, (where $k = 0, 1, \cdots n$).
2) The functions $P_k(t)$ must be complementary solutions of a linear differential equation with constant coefficients if $d/dt\ P_k(t)\epsilon G$.

The $\alpha_1, \alpha_2, \cdots, \alpha_{2m}$ are the roots of the characteristic $A(-p^2) = 0$, and $u(t)$ and $u(t - T)$ are unit step functions which cause $W(t)$ to be zero outside the region where $0 \leqq t \leqq T$.

To determine the solution for $W(t)$ it is required to solve the differential equations

$$M_k(t) = \frac{R(p)^2}{A(-p^2)} P_k(T_0 - t) \qquad k = 0, 1, 2, \cdots n.$$

The function $R(p)^2/A(-p^2)$ is a system response relating $M_k(t)$ as the output to $P_k(T_0 - t)$ as input.

The solution for $W(t)$ obtained from Zadeh and Ragazzini[1] is

$$W(t) = [u(t) - u(t - T)]\Big\{\sum_{k=0}^{n} A_kt^k$$

$$+ \frac{1}{2\pi} R(p) \int_{-\infty}^{+\infty} \frac{S_M(\omega^2)K(j\omega)R(-j\omega)e^{j\omega t}}{A(\omega^2)}\, dw$$

$$+ \sum_{j=1}^{2m} B_je^{\alpha_j t}\Big\} + \sum_{j=1}^{l-m} C_j\delta^{j-1}(t)$$

$$+ \sum_{j=1}^{l-m} D_j\delta^{j-1}(t - T). \tag{33}$$

The two solutions, (32) and (33), differ in two respects. The term

$$\sum_{k=0}^{n} \lambda_k' \frac{[R(p)^2]}{A(-p^2)} P_k(T_0 - t) \tag{34}$$

is substituted for $\sum_{k=0}^{n} A_kt^k$ and the constant C_0 is added.

The complete class of functions G satisfying the con-

[4] For the symbol ϵ, read "belongs to."

ition $DG \epsilon G$ are the set of functions $P_k(t)$ which are ossible solutions of an arbitrary homogeneous linear ifferential equation with constant coefficients. This class composed of arbitrary polynomials, sums of exponentials, ıms of sines and cosines, and products of the above ınctions such as to be a term in the solution of a homo-eneous differential equation with constant coefficients.

For the class of functions in G

$$\sum_{k=1}^{n} \lambda_k' \frac{[R(p)]^2}{A(-p^2)} P_k(T_0 - t) = \sum_{k=1}^{n} \lambda_k'' P_k(T_0 - t). \tag{35}$$

Application of solution to white noise:

Let $M(t) = 0$, $k(\tau)$ not specified

$$\psi_N(t) = \delta(t), \tag{36}$$

nd

$$Q_k = \int_{-\infty}^{+\infty} k(\tau) P_k(T_0 - \tau)\, d\tau.$$

hen (23) becomes

$$\int_0^T W(\tau)[\psi_N(t - \tau)]\, d\tau = \sum_{k=0}^{n} \lambda_k P_k(T_0 - t). \tag{37}$$

ubstituting (36) into (37) gives

$$W(t) = \sum_{k=0}^{n} \lambda_k P_k(T_0 - t). \tag{38}$$

ubstituting in (14) the value of $W(\tau)$ obtained from 38) gives

$$Q_l = \int_0^T \sum_{k=0}^{n} \lambda_k P_k(T_0 - t) P_l(T_0 - t)\, dt$$

$$l = 0, 1, 2, \cdots n.$$

ntroducing the notation

$$S_{kl} = \int_0^T P_k(T_0 - t) P_l(T_0 - t)\, dt = S_{lk}\,, \tag{39}$$

hen

$$Q_l = \lambda_0 S_{0l} + \lambda_1 S_{1l} + \cdots \lambda_n S_{nl} \qquad l = 0, 1, 2, \cdots n$$

$$= S_{l0}\lambda_0 + S_{l1}\lambda_1 + \cdots S_{ln}\lambda_n \tag{40}$$

r in matrix notation

$$Q = S\lambda$$

Q and λ are column vectors and S is an $n \times n$ matrix.

vhere

$$Q = \begin{vmatrix} Q_0 \\ Q_1 \\ \cdot \\ Q_n \end{vmatrix} \qquad \lambda = \begin{vmatrix} \lambda_0 \\ \lambda_1 \\ \cdot \\ \lambda_n \end{vmatrix} \qquad S = \begin{vmatrix} S_{00} & S_{01} & \cdots & S_{0n} \\ S_{10} & S_{11} & \cdots & S_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ S_{n0} & \cdot & \cdots & S_{nn} \end{vmatrix}$$

The necessary and sufficient condition that the deter-ninant S be nonsingular in the interval $0 \leqq t \leqq T$ is hat the functions $P_k(T_0 - t)$ $(k = 0, 1, 2, \cdots n)$ be ineарly independent. Thus it is required that no set of constants E_i exist such that

$$P_k(T_0 - t) = \sum_{\substack{i=0 \\ i \neq k}}^{n} E_i P_i(T_0 - t)$$

where $P_k(t_0 - T)$ is not identically zero in the interval.

The necessary and sufficient condition that the set of functions $P_k(T_0 - t)$ be linearly dependent in the interval is that the Wronskian of the system of functions $P_0(T_0 - t)$, $\cdots P_n(T_0 - t)$ be identically zero in the interval provided the Wronskian of system of functions $P_0(T_0 - t)$, $\cdots$ $P_{n-1}(T_0 - t)$ does not vanish identically.[5]

The Wronskian is defined by the determinant

$$W = \begin{vmatrix} P_0(T_0 - t) & P_1(T_0 - t) & \cdots & P_n(T_0 - t) \\ P_0^{(1)}(T_0 - t) & P_1^{(1)}(T_0 - t) & \cdots & P_n^{(1)}(T_0 - t) \\ \cdot & \cdot & \cdots & \cdot \\ P_0^{(n)}(T_0 - t) & P_1^{(n)}(T_0 - t) & \cdots & P_n^{(n)}(T_0 - t) \end{vmatrix}$$

in the interval $0 \leqq t \leqq T$, where

$$P_i^{(u)}(T_0 - t) \equiv \frac{d^u}{dt^u} P_i(T_0 - t).$$

If the functions $P_k(T_0 - t)$ are found to be linearly dependent, then the system of equations has fewer than $n + 1$ constraints. The linear dependence is eliminated by defining a new set of functions such that the system of constraints will be reduced to a minimum number and the corresponding reduced matrix S will be nonsingular.

Let S^{-1} be the inverse S matrix and assume that S is nonsingular. Then,

$$S^{-1}Q = \lambda$$

From (38) we can write

$$W(t) = P'\lambda$$

where P' is the transpose of P and is given by the row vector

$$[P_0(T_0 - t)P_1(T_0 - t) \cdots P_n(T_0 - t)].$$

Therefore

$$W(t) = P'S^{-1}Q \tag{41}$$

which is a linear function of the input function $P_k(t)$. The mean square error of estimate associated with the optimum filter $W(\tau)$ is given by

$$\sigma^2 = Q'S^{-1}Q \tag{42}$$

where Q' is the transpose of Q. Finally, if the functions $P_k(T - t)$ are orthogonal over the region $0 \leqq t \leqq T$, *i.e.*,

$$\int_{t=0}^{T} P_k(T_0 - t)P_l(T_0 - t)\, dt = \delta_{kl}\,; \tag{43}$$

then S^{-1} is an identity matrix so that

$$\sigma^2 = \sum_{k=0}^{n} Q_k^2. \tag{44}$$

[5] R. Courant, "Differential and Integral Calculus," Nordeman Publishing Co., New York, N.Y., vol. II; 1936.

Appendix I

Let

$$P(t) = \sum_{k=0}^{n} a_k P_k(t) \tag{1}$$

where the a_k are any arbitrary complex numbers not all equal to zero. Let a class of functions G be defined having the following property: $P(t)\epsilon G$ if $DP(t)\epsilon G$. The operator D is a general linear differential equation operator on $P(t)$,

$$D = \sum_{j=0}^{m} b_j \frac{d^j}{dt^j}, \tag{2}$$

where b_j is a set of arbitrary imaginary numbers not all equal to zero. Then the mapping property has the form

$$\sum_{j=0}^{m} \sum_{k=0}^{n} b_j a_k \frac{d^j}{dt^j} P_k = \sum_{k=0}^{n} a_k' P_k(t) \tag{3}$$

where any two of the sets b_j, a_k, or a_k' can be chosen arbitrarily. The problem is to determine the members of the class of G which satisfy the property $DG\epsilon G$. The following lemmas will be required.

Lemma 1

If $\frac{d}{dt} P_k(t)\epsilon G$, then $\frac{d^j}{dt^j} P_k(t)\epsilon G$

where $j = 0, 1, 2, \cdots m$ and $k = 0, 1, 2, \cdots n$.
Proof: Let

$$\frac{d}{dt} P_k(t) = \sum_{u=0}^{n} b_{ku} P_u(t)$$

where b_{ku} is an arbitrary set of constants not all equal to zero and $k = 0, 1, 2, \cdots n$.
Set $d^j/dt^j\ P_k(t) = P_k^{(j)}$. Then in matrix notation, we can write

$$\begin{vmatrix} P_0^{(1)} \\ P_1^{(1)} \\ P_2^{(1)} \\ \cdot \\ \cdot \\ P_n^{(1)} \end{vmatrix} = \begin{vmatrix} b_{00} & b_{01} & \cdot & \cdot & b_{0n} \\ b_{10} & b_{11} & \cdot & \cdot & b_{1n} \\ b_{20} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n0} & b_{n1} & \cdot & \cdot & b_{nn} \end{vmatrix} \begin{vmatrix} P_0 \\ P_1 \\ \cdot \\ \cdot \\ P_n \end{vmatrix}$$

or

$$P^1 = bP. \tag{4}$$

Taking the derivative of both sides of (4) gives

$$| P^{(2)} | = (b)\ | P^{(1)} |, \qquad \therefore\ | P^{(2)} | = (b)^2\ | P |. \tag{5}$$

Repeating the process gives the recursion formula

$$| P^{(k+1)} | = (b)^{(k+1)}\ | P |. \tag{6}$$

Since the matrix $(b)^{(k+1)}$ is a linear transformation on P |, each of the terms $d^j/dt^j\ P_k(t)\epsilon G$.

Lemma 2

The necessary and sufficient condition that $P(t)\epsilon G$ is that

$$\frac{d}{dt} P_k(t)\epsilon G, \quad \text{where} \quad k = 0, 1, 2, \cdots n.$$

Proof:
From (1) and (2) we get

$$DP(t) = \sum_{j=0}^{m} \sum_{k=0}^{n} b_j a_k \frac{d^j}{dt^j} P_k(t). \tag{7}$$

Expanding the right side of (7) gives

$$DP(t) = b_0 \sum_{k=0}^{n} a_k P_k(t) + b_1 \sum_{k=0}^{n} a_k \frac{d}{dt} P_k(t) + \sum_{j=2}^{m} \sum_{k=0}^{n} b_j a_k \frac{d^j}{dt^j} P_k(t). \tag{8}$$

The first term of (8) obviously belongs to G since $AG\ \epsilon\ G$ where A is an arbitrary constant, and the second term belongs in G by the hypothesis. In the third term we can substitute $\sum_{v=0}^{n} u_{kjv} P_v(t)$ for $d^j/dt^j\ P_k(t)$ from (4) of Lemma 1, where u_{kjv} are arbitrary constants. Then the third term also belongs to G since it reduces to a linear sum of $d/dt\ P_k(t)$ where $k = 0, 1, 2, \cdots n$.

Since

$$\sum_{j=0}^{z} B_j G_j \epsilon G \tag{9}$$

where B_j is a set of arbitrary constants and G_j is any particular member of G, it follows that $DP(t)\epsilon G$ when given $d/dt\ P_k(t)\epsilon G$.

To show that the condition is necessary, let

$$\frac{d^j}{dt^j} P_k(t) = \sum_{u=0}^{n} c_{jku} P_u(t) + Q_{jk}(t) \tag{10}$$

where $Q_{jk}(t)\epsilon \bar{G}$ ($\bar{G}$ is the group of all functions not in G). Then if $Q_{jk}(t)\epsilon \bar{G}$, there exists no linear transformation of the form

$$Q_{jk}(t) = \sum_{v=0}^{n} e_{jkv} P_v(t) \tag{11}$$

for all j and k. Substituting (10) into (7) gives

$$DP(t) = \sum_{u=0}^{n} c_u' P_u(t) + \sum_{k=0}^{n} \sum_{j=0}^{m} a_k b_j Q_{jk}(t) \tag{12}$$

where the relationship must hold over all arbitrary constants a_k and b_j (a_k and b_j not all equal to zero). Since

$$\sum_{j=0}^{m} \sum_{k=0}^{n} a_k b_j Q_{jk}(t)\epsilon \bar{G}, \quad \text{then} \quad DP(t)\epsilon \bar{G}.$$

However, $DP(t)\epsilon G$ if and only if

$$\sum_{k=0}^{n} \sum_{j=0}^{m} a_k b_j Q_{jk}(t) \equiv 0.$$

Since a_k and b_j are arbitrary sets of constants,

$$\sum_{j=0}^{m} \sum_{k=0}^{n} a_k b_j Q_{jk}(t) \equiv 0$$

and only if $Q_{jk}(t) \equiv 0$ over all j and k. In particular for $= 1$, $Q_{1k}(t) = 0$ so that the condition $d/dt\ P_k(t) \epsilon G$ is ecessary and sufficient.

Since the condition

$$P^{(1)}(t) = bP \tag{13}$$

necessary and sufficient, the solutions of the system of fferential equations of (13) defines the class of functions

The solution to (13) is obtained thus:[6]

Let

$$y_k = P_k(t)$$

nd

$$\frac{d^u}{dt^u} P_k(t) = p^u y_k.$$

Then (11) can be written as the matrix equation

$$pIy = by \tag{14}$$

here p is a scalar multiplier, I is an $n + 1$ by $n + 1$ dentity matrix, and y is a column matrix. Eq. (14) can e written $(pI - b)\ y = 0$. Let $F(p)$ be the determinant $pI - b)$, and the solution to the system is given by

$$F(p)y_i = 0, \qquad i = 0, 1, 2, \cdots n$$

rovided $F(p)$ is not identically zero. Since $F(p)$ is a olynomial in p, the $y_i = P_i(t)$ are the complementary olutions of linear differential equations with constant oefficients.

Thus the complete class of functions $P(t) \epsilon G$ is the undamental system

$$P(t) = \sum_{k=0}^{n} a_k P_k(t)$$

vhere

$$F(p)P_k(t) = 0 \qquad k = 0, 1, 2, \cdots n.$$

The complete class of functions that belong to G consists of arbitrary polynomials, sums of exponentials, ums of sines and cosines and products of these functions vhich could be terms in the solution of a homogeneous inear differential equation with constant coefficients. For example, the function $P(t) = \sum_{k=0}^{n} a_k t^k$ is a solution o the differential equation

$$\frac{d^{n+1}}{dt^{n+1}} P(t) = 0$$

r

$$P(t) = \sum_{k=0}^{n} a_k e^{dkt}$$

s the solution of a linear constant coefficient differential perator D_{n+1}, or order $n + 1$ operating on $P(t)$. *i.e.*, $)_{n+1}P(t) = 0$ where the roots of the characteristic equaion $D_{n+1}(p) = 0$ are $d_0, d_1, d_2, \cdots d_n$, and the roots are ll distinct.

[6] E. L. Ince, "Ordinary Differential Equations," Dover Publiations, New York, N.Y.; 1944.

Appendix II

Modification of Solution for $W(t)$ When the Minimization is with Respect to the Entire Time Interval[7]

The quantities Q_k defined by (14) are constants since the error is minimized at a particular instant of time. In a more general situation where the error is to be minimized for all t the Q_k are functions of time, $Q_k = Q_k(t)$. Thus — the Lagrangian multiplers become functions of time, and the integral (23) assumes the form.

$$\int_0^T W(\tau)[\phi_M(t - \tau) + \phi_N(t - \tau)]\, d\tau = \sum_{k=1}^{n} \int_{-\infty}^{+\infty} \lambda_k(\tau) P_k(\tau - t)\, d\tau + \int_{-\infty}^{+\infty} k(\tau)\phi_M(t - \tau)\, d\tau$$

The solution of this integral equation, together with the constraint (14) (with t substituted for T_0) yields $W(t)$ and the Lagrangian multipliers $\lambda_1(t)$, $\lambda_2(t)$, $\cdots$ $\lambda_n(t)$. The solution for this case is more difficult than that of solving the single integral (23).

There is, however, a reasonably wide class of functions $P_k(t)$ for which the above system reduces to a single integral equation. This class, F, comprises those functions $P_k(t)$ which have the property that $P_k(t - \tau)$ admits of representation in the form.

$$P_k(t - \tau) = \sum_{M=1}^{m} a_M^k(\tau)\psi_M^k(t)$$

where the $\psi_M^k(t)$, $M = 1, 2, m$, are linearly independent functions. In this case, (14) reduces to the set of equations

$$\int_0^T a_M^k(\tau)W(\tau)\, d\tau = \int_{-\infty}^{+\infty} a_M^k(\tau)k(\tau)\, d\tau = R_M^k$$

where the R_M^k are constants, and the system of integral equations degenerates into the single integral equation

$$\int_0^T W(\tau)[\phi_M(t - \tau) + \phi_N(t - \tau)]\, d\tau = \sum_k \sum_u \lambda_u^k a_u^k(t) + \int_{-\infty}^{+\infty} k(\tau)\phi_M(t - \tau)\, d\tau$$

in which the Lagrangian multipliers λ_M^k are constants.

Utilizing the above development and under the same assumptions from which (32) is derived one obtains the modification of the solution for $w(t)$ by replacing

$$\sum_{k=0}^{n} \lambda_k' \frac{[R(p)]^2}{A(-p^2)} \{P_k(T_0 - t)\}$$

by

$$\sum_{k=0}^{n} \sum_{M=1}^{m} \lambda_M^k \frac{[R(p)]^2}{A(-p^2)} \{a_M^k(t)\}.$$

[7] The development of the integral equation is due to the referee. Note that $G \epsilon F$.

The Correlation Function of a Sine Wave Plus Noise after Extreme Clippings

J. A. McFADDEN†

Summary—**This paper presents a simple formula for the correlation function of an extremely clipped signal when the input is Gaussian noise plus a sine wave of small amplitude.**

CONSIDER two signals x_1 and x_2 at the respective times t and $t + \tau$. Let

$$\begin{aligned} x_1(t) &= y_1(t) + A \sin \omega t, \\ x_2(t + \tau) &= y_2(t + \tau) + A \sin \omega(t + \tau), \end{aligned} \tag{1}$$

where A is a constant. $y_1(t)$ and $y_2(t + \tau)$ are noise fluctuations, obeying the bivariate normal distribution with zero means, equal variances σ^2, and with a (normalized) correlation $\rho(\tau)$ [*i.e.*, the expected value of the product $y_1(t)y_2(t + \tau)$, divided by the variance].

By an elementary statistical calculation we can show that the (normalized) correlation between $x_1(t)$ and $x_2(t + \tau)$ is

$$R(\tau) = \frac{\rho(\tau) + a^2 \cos \omega\tau}{1 + a^2}, \tag{2}$$

where $a^2 = A^2/2\sigma^2$, the signal-to-noise power ratio. (This calculation involves an ensemble average for the noise and a time average over one cycle of the sine wave.) In general, $R(\tau)$ is called the cross-correlation function of $x_1(t)$ and $x_2(t)$, and $\rho(\tau)$ is the cross correlation of $y_1(t)$ and $y_2(t)$. If $y_1(t)$ and $y_2(t)$ are identical, then $R(\tau)$ is the autocorrelation function of $x_1(t)$ and $\rho(\tau)$ is the autocorrelation of $y_1(t)$.

Now suppose that $x_1(t)$ and $x_2(t)$ are extremely clipped; *i.e.*, let the outputs be $\xi_1(t)$ and $\xi_2(t)$, where

$$\begin{aligned} \xi_i(t) &= 1 \quad \text{when} \quad x_i(t) \geq 0, \\ &= -1 \quad \text{when} \quad x_i(t) < 0, \end{aligned} \tag{3}$$

where $i = 1, 2$. When no sine wave is present, $A = 0$ and $R(\tau) = \rho(\tau)$, and the correlation between $\xi_1(t)$ and $\xi_2(t + \tau)$ is given by[1]

$$r(\tau) = (2/\pi) \sin^{-1} R(\tau). \tag{4}$$

When no noise is present, $\sigma^2 = 0$ and $R(\tau) = \cos \omega\tau$; in this case (4) is also valid, the function $r(\tau)$ being a saw-tooth wave. This case can be calculated by elementary methods or by considering a sine wave as the limiting case of extremely narrow-band noise.

When a^2 is not zero or infinite, (4) does not hold. The purpose of this note is to derive an approximate correction for small signal-to-noise ratios, *i.e.*, for $a^2 \ll 1$.

Davenport,[2] following Rice and Middleton, has given a solution for $r(\tau)$ in the general case. His solution is a double series expansion in terms of $\rho^k(\tau) \cos m\omega\tau$, with coefficients involving confluent hypergeometric functions of a^2. The present solution, for the restricted case $a^2 \ll 1$, is a power series in a^2, the coefficients being elementary functions of $\rho(\tau)$ and $\omega\tau$. This result should be better adapted for computation than Davenport's, especially when $\rho(\tau)$ is near unity.

Although this approximation can be derived from Davenport's double series, the following method is simpler: First, for a given value of t, we shall calculate P_{++}, the probability that $x_1(t)$ and $x_2(t + \tau)$ are both positive. Second, we shall obtain the average of P_{++} over one complete cycle of ωt. Let this average be $\overline{P}_{++}$. Finally we shall obtain $r(\tau)$, the correlation between the outputs $\xi_1(t)$ and $\xi_2(t + \tau)$, as a function of $\overline{P}_{++}$.

First let us calculate P_{++}. Under the bivariate normal distribution, the probability that $y_1 > -z_1$ and $y_2 > -z_2$ is[3]

$$F(z_1, z_2) = \frac{1}{2\pi\sigma^2(1 - \rho^2)^{1/2}} \cdot \int_{-z_1}^{\infty} \int_{-z_2}^{\infty} \exp\left\{-\frac{y_1^2 + y_2^2 - 2\rho y_1 y_2}{2\sigma^2(1 - \rho^2)}\right\} dy_1\, dy_2. \tag{5}$$

By differentiation with respect to z_1 and z_2, we may expand $F(z_1, z_2)$ in a double Maclaurin series,

$$\begin{aligned} F(z_1, z_2) = F(0, 0) &+ \left(\frac{\partial F}{\partial z_1}\right)_{0,0} z_1 + \left(\frac{\partial F}{\partial z_2}\right)_{0,0} z_2 \\ &+ \frac{1}{2!}\left[\left(\frac{\partial^2 F}{\partial z_1^2}\right)_{0,0} z_1^2 + 2\left(\frac{\partial^2 F}{\partial z_1\, \partial z_2}\right)_{0,0} z_1 z_2 \right. \\ &\left. + \left(\frac{\partial^2 F}{\partial z_2^2}\right)_{0,0} z_2^2\right] + \cdots . \end{aligned} \tag{6}$$

For example, two of the coefficients are

$$\begin{aligned} F(0, 0) &= \tfrac{1}{4} + (1/2\pi) \sin^{-1} \rho, \\ \left(\frac{\partial^2 F}{\partial z_1\, \partial z_2}\right)_{0,0} &= 1/[2\pi\sigma^2(1 - \rho^2)^{1/2}]. \end{aligned} \tag{7}$$

If we substitute $z_1 = A \sin \omega t$ and $z_2 = A \sin \omega(t + \tau)$, then $F(z_1, z_2)$ becomes P_{++}, the probability that $x_1(t)$ and $x_2(t + \tau)$ are both positive at a given time t. Only the even terms of (6) need be considered, since the odd ones will integrate to zero over a complete cycle of ωt.

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[1] See, for example, J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," McGraw-Hill Book Co., Inc., New York, N.Y., p. 58; 1950.

[2] W. B. Davenport, Jr., "Signal-to-noise ratios in bandpass limiters," *J. Appl. Phys.*, vol. 24, p. 720 (13); 1953.

[3] H. Cramér, "Mathematical Methods of Statistics," Princeton Univ. Press, Princeton, N.J., p. 290; 1946.

To obtain $\overline{P}_{++}$, we integrate $P_{++}\, d(\omega t)$ from $\omega t = 0$ to $= 2\pi$ and divide by 2π. $\overline{P}_{++}$ represents the average action of time in which $x_1(t)$ and $x_2(t + \tau)$ are simultaneously positive.

Next we must obtain $r(\tau)$, the correlation between $\xi_1(t)$ and $\xi_2(t + \tau)$. The mean values of both signals are zero. Moreover the two signals can only be $+1$ or -1; therefore the variances are unity and the correlation $r(\tau)$ is simply the expected value $E[\xi_1(t)\xi_2(t + \tau)]$. To obtain this average we need consider only four possibilities: $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, and $(-1, -1)$. Let the probability of each combination be $\overline{P}_{++}$, $\overline{P}_{+-}$, etc., then

$$r(\tau) = (+1)\overline{P}_{++} + (-1)\overline{P}_{+-} + (-1)\overline{P}_{-+} + (+1)\overline{P}_{--}. \quad (8)$$

By symmetry, $\overline{P}_{++} = \overline{P}_{--}$ and $\overline{P}_{+-} = \overline{P}_{-+}$. Furthermore the sum of all four probabilities is unity. Then we conclude that

$$r(\tau) = 4\overline{P}_{++} - 1, \quad (9)$$

and this relation is the last step in the required calculation.

If we carry out the foregoing operations, we find for the correlation,

$$= \frac{2}{\pi}\left[\sin^{-1}\rho + a^2\frac{\cos\varphi - \rho}{(1-\rho^2)^{1/2}} - \frac{a^4}{4}\frac{(\cos\varphi - \rho)(2 - \rho\cos\varphi - \rho^2)}{(1-\rho^2)^{3/2}} + O(a^6)\right], \quad (10)$$

where $\varphi = \omega\tau$. In terms of R, given by (2), the expression is even simpler. We have

$$= (2/\pi)[\sin^{-1} R + \tfrac{1}{4}a^4(1-\rho^2)^{-3/2} \cdot(\cos\varphi - \rho)(2 - \rho\cos\varphi - \rho^2) + O(a^6)]. \quad (11)$$

When a^2 vanishes we have the limiting case (4). Thus (11) provides a correction to (4) when a^2 is small. This correction is of fourth order in a. To second order, (4) is still valid.

When $r(\tau)$ is an autocorrelation, $\rho(0) = 1$ and $r(0)$ should also be unity. This requirement is indeed satisfied, at least to the fourth order. The limiting form of (11) when $\tau \to 0$ appears indeterminate, but it can be evaluated provided $\rho'(0) = 0$ and $\rho''(0)$ exists. These conditions are usually fulfilled for the power spectra which we might assume for the noise.

One application of the above result is to polarity-coincidence correlators, which can be used for signal detection.[4]

[4] J. J. Faran, Jr. and R. Hills, Jr., "Correlators for Signal Reception," Harvard Univ., Acoustics Res. Lab. Tech. Memo. 27, p. 8; 1952.

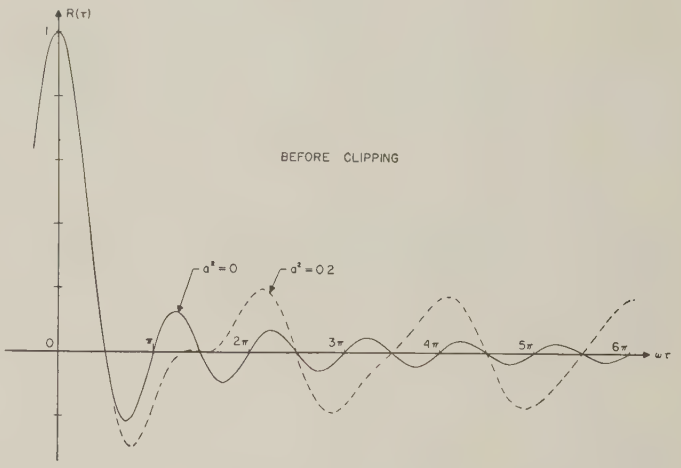

Fig. 1—The autocorrelation function $R(\tau)$ of a sine wave plus Gaussian noise (before clipping) when the sine wave has a frequency ω, the noise has an ideal low-pass band from 0 to 2ω, and the signal-to-noise power ratio is a^2.

Fig. 2—The autocorrelation function $r(\tau)$ of a sine wave plus Gaussian noise after extreme clipping, when the input signal has the same spectrum as in Fig. 1.

Examples of autocorrelation functions are shown in Fig. 1 and Fig. 2. Suppose the spectral density of the noise is constant between 0 and 2ω and zero elsewhere. Then $\rho(\tau) = \sin 2\omega\tau/2\omega\tau$, and this function is given by the solid curve in Fig. 1. The broken line in Fig. 1 shows the autocorrelation function $R(\tau)$ after the addition of a sine wave with frequency ω, when the signal-to-noise power ratio is $a^2 = 0.2$. $R(\tau)$ was computed from (2). Fig. 2 shows the corresponding autocorrelation functions after extreme clipping, computed from (11). The functions after clipping are quite similar to those before clipping except for the occurrence of a triangular peak at $\tau = 0$.

A Note on Two Binary Signaling Alphabets

DAVID SLEPIAN†

Summary—**A generalization of Hamming's single error correcting codes is given along with a simple maximum likelihood detection scheme. For small redundancy these alphabets are unexcelled. The Reed-Muller alphabets are described as parity check alphabets and a new detection scheme is presented for them.**

Introduction

CERTAIN general properties of parity check alphabets for use on the binary symmetric channel were investigated in a recent paper by the author.[1] In particular, it was shown that maximum likelihood detectors for such alphabets assume a particularly simple form. In addition certain optimal small-sized alphabets were presented.

This paper describes two simple families of binary-signaling alphabets and detectors that are easy extensions of the material contained in that earlier paper.[1] Both families consist of parity check alphabets and both contain alphabets of arbitrarily large size. Members of the first are designed for use in situations requiring little redundancy, and for certain ranges of the pertinent parameters these alphabets cannot be excelled. The rate for these alphabets approaches unity with increasing alphabet size, however, so that they cannot be used to approach the Shannon rate in nontrivial cases. It is not known whether members of the second family can be used to approach the Shannon rate.

As in the reference cited,[1] we consider communication over a binary symmetric channel by means of parity check alphabets having k information places, p check places, and $n = p + k$ places all told. That is, if $a_1a_2 \cdots a_kb_1b_2 \cdots b_p$ is a typical letter of the signaling alphabet, where the a's and b's are either zero or one, then $a_1 \cdots a_k$ can be chosen arbitrarily and

$$b_j = \sum_{1}^{k} \cdot c_{ij}a_i, \qquad j = 1, 2, \cdots, p \tag{1}$$

where the dot denotes summation modulo 2. The kp entries of the matrix $C = (c_{ij})$ determine the alphabet in question.

Generalized Hamming Alphabets

If p and k are chosen so that

$$k \geq 2^p - \sum_{i=0}^{3} \binom{p}{i},$$

then a simple parity check alphabet and maximum likelihood detector can be described which cannot be excelled by any other alphabet of 2^k letters using n-digit binary sequences. That is, no other alphabet of the same size has a smaller average probability of error per letter. These small redundancy alphabets are generalizations of Hamming's original single error correcting code.[2]

The C matrix for these alphabets is constructed as follows. The first row contains p ones. Successive rows of the matrix are obtained by writing in any order all the $\binom{p}{1}$ p-place binary sequences with exactly 1 zero, then in any order all the $\binom{p}{2}$ p-place binary sequences with exactly 2 zeros, etc., until k rows have been written. If c_{ij} is the element in the ith row and jth column of this kxp matrix, then the parity check rules are given by (1).

The matrix C just described also serves as a code book for the maximum likelihood detector for this alphabet. For each received (possibly erroneous) letter, form the parity check sequence $f_1f_2 \cdots f_p$ where

$$f_j = e_j + \sum_{i=1}^{k} \cdot c_{ij}d_i, \qquad j = 1, 2, \cdots, p$$

and where d_i is the binary symbol in the ith information place of the received letter and e_j is the binary symbol in the jth check position of the received letter. If $f_1f_2 \cdots f_p$ is the same as the rth row of C, the binary digit in the rth information place of the received letter should be altered. If $f_1f_2 \cdots f_p$ is not listed among the rows of C and does not contain exactly three 1's, the binary digits in those check places of the received letter should be changed that correspond to the places of $f_1f_2 \cdots f_p$ that have the binary digit 1. If the parity check sequence $f_1 \cdots f_p$ has exactly three 1's and does not appear as a row of C, a row of C having four 1's is located such that three of these four 1's are in the same places as the three 1's of the parity check sequence. Let this row of C be the ith row and let the 1 in this row that does not correspond to a one in the parity check sequence be located in the jth column. Then in the received letter, the ith information place and the jth check place should be altered.

The alphabets and detectors just described correct 2^p-1 single errors if $n > 2^p-1$. If

$$2^p - \sum_{0}^{3} \binom{p}{i} \leq n \leq 2^p - 1,$$

the alphabets correct all $\binom{n}{i}$ single errors and in addition 2^p-1-n double errors.

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[1] D. Slepian, "A class of binary signaling alphabets," *Bell Sys. Tech. J.*, vol. 35, pp. 203–234; January, 1956.

[2] R. W. Hamming, "Error detecting and error correcting codes," *Bell Sys. Tech. J.*, vol. 29, pp. 147–160; April, 1950.

Proof of the above statements follows readily from Slepian.[3] The description of the generalized Hamming alphabets presented here is simpler than that given by him. The detection rule has not been given before.

Reed-Muller Alphabets

The Reed-Muller alphabets[4] can also be described as parity check alphabets. Each letter has $n = 2^m$ binary places of which

$$k = \sum_{i=0}^{r} \binom{m}{i}$$

are information places and $p = n - k$ are check places. For each non-negative integer value of m and $r \leq m$ there is one Reed-Muller alphabet.

It is convenient to label the entries in the k information places of a typical letter of an alphabet by the symbols $a_o, a_1, a_2, \cdots, a_m, a_{12}, a_{13}, \cdots, a_{(m-1)m}, a_{123}, \cdots, a_{123\ldots r}, \cdots, a_{(m-r+1)(m-r+2)\ldots m}$. That is, the a's are labelled by all k combinations of the integers 1, 2, $\cdots$, m taken r or fewer at a time. Similarly, the check places, $b_{12\ldots(r+1)}$. $b_{12\ldots r(r+2)}, \cdots, b_{12\ldots m}$, are labelled by all $2^m - k$ combinations of the integers 1, 2, $\cdots$, m taken $r + 1$ or more at a time. It will be convenient to denote a typical a or b with j subscripts by $a_\alpha^{(j)}$ or $b_\alpha^{(j)}$ respectively and to write $\alpha \subset \beta$ for two subscripts α and β if all the numerals in α are also present in β. The special subscript o is to be considered as the empty set and is contained in every β.

Let D_{ij}^r be one or zero accordingly as $\binom{j-i-1}{r-i}$ is odd or even respectively. The parity check rules for the Reed-Muller alphabets can then be written

$$b_\beta^{(j)} = \sum_{i=0}^{r} \cdot D_{ij}^r \sum_{\alpha \subset \beta} \cdot a_\alpha^{(i)} \tag{2}$$

$j = r + 1, r + 2, \cdots m$, all possible β.

Here all sums are modulo 2 and the α sum is over all indexes α that are i-fold combinations of the j integers that comprise β; *e.g.*, if $\beta = 1346$, the α sum is for $i = 2$, $\sum_{\alpha \subset \beta} \cdot a_\alpha^{(2)} = a_{13} \dot{+} a_{14} \dot{+} a_{16} \dot{+} a_{34} \dot{+} a_{36} \dot{+} a_{46}$.

A maximum likelihood detector for the Reed-Muller alphabets can be constructed in a straightforward manner.[1] No simplification in the detector has been found that depends on the special nature of the Reed-Muller alphabet. For large m and r the labor involved in constructing the detector is considerable.

An important feature of the Reed-Muller alphabets is that every two letters of an alphabet with parameters m and r differ in at least 2^{m-r} places. This implies that a detector can be built which will correct all errors in a sent letter provided these errors be $l = 2^{m-r-1} - 1$ or fewer in number. Such a detector can be constructed as follows. First a code book is made which associates with each n place binary sequence $z_o, z_1, \cdots, z_{12}, \cdots, z_{12\ldots m}$ having l or fewer ones an $(n - k)$-place parity check sequence $f_{12\ldots r}, \cdots, f_{12\ldots m}$. The f's are given by

$$f_\beta^{(j)} = z_\beta^{(j)} \dot{+} \sum_{i=0}^{r} \cdot D_{ij}^r \sum_{\alpha \subset \beta} \cdot z_\alpha^{(i)}, \tag{3}$$

$j = r + 1, r + 2, \cdots, m$, all possible β.

The code book contains $\sum_{i=0}^{l} \binom{n}{i}$ pairs of entries. For each received (possibly erroneous) letter, an $(n - k)$-place parity check sequence is formed as in (3) where the digits of the received letter are substituted for the z's. If the parity check sequence is not in the code book, the detector makes no decision as to the sent letter. If the parity check sequence is in the code book, the corresponding z entry having l or fewer ones is found. Those places of the received (possibly erroneous) letter are altered that correspond to places of the z entry that have the digit 1.

It is not difficult to show that if the parity check sequence has 1 or fewer 1's, the detector prescribes that no changes be made in the information places of the received letter. This results in a simplification of the detector when it is only desired to make corrections in the information places. The code book only needs to be constructed for those z sequences that have at least one l in their information places.

The description of the Reed-Muller alphabets given here is new as is the detector described. In some cases this detector is simpler than that given.[3]

The parity check rules (2) for the Reed-Muller alphabets can be derived as follows. Consider the m-row by 2^m column array of the components of the vectors $x_1, x_2, \cdots, x_m$ as given by Reed's.[4] The successive columns of this array are a listing of the integers from 0 to $2^m - 1$ in usual binary notation. The columns of this array can be labelled with sets of integers or α-symbols as used in (2) of this note. In particular, the column having 1's in rows $i_1, i_2, \cdots i_j$ is labelled by the α symbol $i_1 i_2 \cdots i_j$. The column containing no 1's is labelled by the special symbol o.

Permute the columns of the array so that the labels are in the *natural* order o, 1, 2, $\cdots$, m, 12, 13, $\cdots$, 123 $\cdots$ m. Consider the rows of this array as vectors and denote the ith row by y_i. Vectors whose subscripts are α symbols can now be formed from these y's. If α is the collection of integers $i_1 i_2 \cdots i_j$, then the βth component of y_α is the product of the βth components of $y_{i_1}, y_{i_2}, \cdots, y_{i_j}$. The vector y_o has all its components unity.

The vectors $y_0, y_1, \cdots, y_{12}, \cdots, y_{123}$, etc., correspond respectively to Reed's vector $I, x_1, \cdots, x_1x_2, \cdots, x_1x_2x_3$, etc. The y vectors have the distinguishing feature that the component of y_α with label β is unity if and only if $\alpha \subset \beta$.

[3] Slepian, *op. cit.*, paragraph 27.

[4] I. S. Reed, "A class of multiple error-correcting codes and the decoding scheme," Trans. IRE, vol. IT-4, pp. 38–49; September, 1954.

Reed's[4] (5) shows that the Reed-Muller alphabets are obtained by forming all possible linear combinations (modulo 2) of the first k y-vectors. The same alphabet will be obtained if all linear combinations are formed of a set of k vectors w_α that are linearly independent combinations of the k y_α's. The following is such a collection of vectors:

$$w_\alpha = \sum_{\beta \supset \alpha} \cdot y_\beta \tag{5}$$

α composed of r or fewer integers,
β composed of r or fewer integers.

From (4) and (5) it follows that the component of w_α labelled μ will be 1 if and only if there are an odd number of subscript symbols β such that $\alpha \subset \beta$ and $\beta \subset \mu$. If $\alpha \not\subset \mu$, then, the μth component of w_α will be zero.

It is convenient to treat the case when $\alpha \subset \mu$ in two parts.

1) If μ is comprised of $j \leq r$ integers and α is comprised of $i \leq j$ integers, then there are 2^{j-i} β symbols such that $\alpha \subset \beta \subset \mu$. Since 2^{j-i} is even except when $j = i$, this shows that the first k components of w_α are zero with the single exception of the component labelled α which has the value 1. The w_α are therefore linearly independent.

2) If μ is comprised of $j > r$ integers and α is comprised of $i \leq r$ integers, then there are

$$\binom{j-i}{0} + \binom{j-i}{1} + \cdots + \binom{j-i}{r-i}$$

β symbols of r or fewer integers such that $\alpha \subset \beta \subset \mu$. From

$$\binom{j-i}{\nu} = \binom{j-i-1}{\nu} + \binom{j-i-1}{\nu-1}$$

it follows that

$$\sum_0^{r-i} \binom{j-i}{\nu} = \sum_0^{r-i-1} 2\binom{j-i-1}{\nu} + \binom{j-i-1}{r-i}$$

or

$$\sum_0^{r-i} \binom{j-i}{2} \equiv \binom{j-i-1}{r-i} \mod 2.$$

Therefore, when μ is comprised of more than r integers, the component of w_α labelled μ is 1 if and only if $\alpha \subset \mu$ and $\binom{j-i-1}{r-i}$ is odd.

Consider the array of n columns and k rows whose entry in the row labelled α and column labelled μ is the component of w_α labelled μ. From the foregoing it is seen that the first k rows and k columns of this array consist of the $k \times k$ unit matrix. The last $n - k$ columns are linear combinations of the first k columns. Indeed, it is easily seen that the column labelled μ is the mod 2 sum of those columns whose labels are the names of the rows that have a 1 in the column labelled μ. This statement, combined with the preceeding paragraph, establishes the parity check rules (2).

The detector described for the Reed-Muller alphabets follows trivially from the results obtained by Slepian.[1]

Generating a Gaussian Sample*

S. STEIN† AND J. E. STORER‡

Summary—**The general theoretical difficulties in analyzing the effect of a random input signal on a known system are pointed out. Basically, if certain output statistics are computed directly, each statistic represents a complete, separate problem. An alternative analytical computational procedure is suggested, using a Monte Carlo type technique in which the output is obtained by numerical integration from sequences of values which represent members of the statistical ensemble of the input process. For such applications, or for other possible uses such as in testing, it is necessary to generate statistical sequences, analogous to tables of random numbers.**

Techniques are discussed for analytically generating such sequences, to correspond to gaussian probability distributions which are further characterized by arbitrarily specified power spectra or autocorrelation functions. The procedure makes use of the standard tables of random numbers, these numbers being distributed uniformly and without correlation. The exact statistical generation of N values of a sequence is shown to require, in general, the diagonalization (or solution for the eigenvalues and eigenvectors) of an Nth order matrix; two simpler approximate procedures are also described.

Introduction

IN THE COURSE of a recent investigation, the problem arose of determining the effect of a nonlinear system upon a signal which could only be specified statistically. The input-output relationship was resolved into the form of an integral, where however, the integrand was not linearly related to the random input process. Therefore, despite the fact that the input process was gaussian and random, it was impossible to obtain predictions on the probability distribution of the output. It was found impossible, moreover, to compute analytically even the simpler moments of the output. Hence no information was available on an analytical basis.

As a consequent attempt to resolve the problem, numerical integration was considered for the multiple integrals which described the moments. These computations were, however, estimated to be much too lengthy and time consuming, even for setting up on a large scale computer. Nevertheless, the concept of numerical integration did serve to bring forth another idea, which is somewhat similar in philosophy to the Monte Carlo technique. Namely it was suggested that the original integral relating the input and output process should be used, and that a numerical integration should be carried out in which the input process is represented by its sample values at the appropriate intervals. By carrying out the integration for many values of the independent variable (usually time) of the output process, successive sample values of the output process would be obtained. The latter sequence or several such sequences could then be analyzed statistically, to obtain a statistical picture of the output, even including probability distributions. Obviously a simple procedure of the same sort could be used to obtain a description of a random process $y(t)$, related to some other known random process $x(t)$, by any type of functional relation

$$y(t) = F[x(t)] \tag{1}$$

where F in general may not be a linear process. The only limitations on these procedures are that 1) sufficiently short sampling intervals must be used, *i.e.*, intervals appreciably shorter than the correlation time of any process involved; 2) the length of the sequences obtained, or the number of sequences calculated out, must be sufficient to give reliable statistics (the rms error generally varying inversely as the square root of the number of values taken in the averaging); and 3) it is necessary to know a sequence of sampled values of the input process, which obey the proper statistics. In general, however, the statistics are the known quantity, while sample sequences of the process are not necessarily available.

While an experimental approach to obtaining such sequences may sometimes be feasible, it is not always possible. Therefore, it becomes desirable to consider whether a set of random values could be generated analytically to satisfy given statistics. Such sets of values could obviously find application in fields far removed from the solution of functional relationships, for example in such fields as applied psychology. Since most random processes originate as gaussian processes, and any further modification of the members of the ensemble can be taken into account by direct computation, the investigation was confined to a study of gaussian processes, which are characterized completely in terms of their autocorrelation function or power spectrum. The feasibility of analytical generation of sample values to correspond to given gaussian statistics is the subject of the remainder of this paper.

Generating an Uncorrelated ("White Noise") Gaussian Random Process

An uncorrelated gaussian process is completely described by its first-order probability distribution

$$W_1(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \tag{2}$$

where $\bar{x} = 0$ has been assumed (there will be no loss in generality). It is desired to choose successive, independent

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values of x, conforming to this distribution for a given value of σ.

It is first necessary to consider the property of randomness. It is apparent, after a little thought, that probably the only reasonable way to incorporate this property is to somehow make use of a sequence of purely random numbers, such as are available in random number tables. The successive values of x must then be related to the successive values of another quantity, r, chosen from a sequence of random numbers. The sequence of values, r, can be considered to be a random stationary process, with a distribution uniform over (0, 1). Fortunately, the required transformation is simple, and well-known; it is obtained readily from the requirement that $W_1(r)\,dr = W_1(x)\,dx$, where $-\infty < x < \infty$, $0 < r < 1$.

The appropriate solution to this differential equation is readily shown to be given in terms of the error-function as

$$r = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]. \tag{3}$$

For each value of r selected from a random number sequence, it is merely necessary to invert this relationship in order to obtain a set of values of x which conform, statistically, to the required distribution [3].

Shaping by a Filter

The next problem is to extend the procedure so as to generate sequences of correlated numbers, *i.e.*, sample values from a gaussian random process having a power spectrum of finite width. Perhaps the first thought which might occur to those in electronics is to accomplish this as the analytical equivalent of passing the signal through a shaping filter, having the desired spectrum.

Thus, with the set of values of the white noise given as x_k, a set of values y_k would be computed at intervals of τ, by the formula

$$y_k = \sum_j h(k\tau - j\tau)x_j. \tag{4}$$

Here $h(t)$ is the filter response function. It should be noted that (4) is mathematically only a linear transformation. This thought will be considered again shortly, but it should be recalled here that any such transformation preserves the stationary gaussian random character of the process. If the coefficients are interpreted in terms of the filter concept, where $h(t)$ is defined to be zero when $t < 0$, the summation will actually have a finite upper index, at $j = k$. In addition, there will always be a finite lower limit to the summation, corresponding to the fact that only a finite number of terms will be summed. It would seem intuitively desirable, and appears to lead to desirable effects, if the same number of x_j values are used in computing each y_k. Thus, summation over $N + 1$ terms is properly written as

$$y_k = \sum_{j=k-N}^{k} h(k\tau - j\tau)x_j. \tag{5}$$

There is, of course, a natural question as to the accuracy which may be obtained in obtaining values for the y-process by this method. A preliminary error analysis, and some sample computations which have been made, seem to show that it is not an unreasonable procedure; it is particularly attractive because of the computational ease as compared with the more exact procedure to be described next.

The Multidimensional Gaussian Process

There is a more mathematical manner of looking at the problem of generating samples to fit a gaussian process with a given autocorrelation function. This is to consider that if N sample values are required, then the Nth order joint distribution density of this set of N samples, averaged over an ensemble of such sets, is specified by the corresponding Nth order gaussian probability distribution. That is, the y_k are now to be generated randomly so that they satisfy

$$W_N(y_1, \cdots y_N) = \frac{1}{(2\pi)^{N/2}} \frac{1}{|\mu|^{1/2}} \cdot \exp\left\{-\frac{1}{2}\sum_{k,m=1}^{N} (\mu^{-1})_{km} y_k y_m\right\}. \tag{6}$$

Here, the y_i's are assumed to occur at times t_i, and the desired autocorrelation function, $R(t_i - t_j) = \overline{y_i y_j}$, is contained implicitly in the $N \times N$ autocorrelation matrix, μ, defined by

$$(\mu)_{ij} = R(t_i - t_j), \tag{7}$$

with $|\mu|$ the determinant of the matrix, and μ^{-1} its inverse.

It is immediately apparent that one way to generate such a process, using information already available, is to seek a linear transformation connecting the y-process to an x-process in which successive values are uncorrelated [1, 2]. This is mathematically completely equivalent to seeking the transformation which diagonalizes the quadratic form in (6), a technique well known in matrix algebra. With this transformation, the x-process obtained will satisfy an Nth order joint probability distribution in the form

$$W_N(x_1, \cdots x_N) = \text{Const.} \exp\left\{-\frac{1}{2}\sum_{k=1}^{N} \frac{x_k^2}{\sigma_k^2}\right\} \tag{8}$$

where each of the x_k can thus be considered as statistically independent, and hence can be generated by the procedure introduced earlier in the second section.

The diagonalization of the μ matrix, it will be recalled, is equivalent to finding the eigenvalues and eigenvectors of the matrix μ, and will yield a result such that $(P^t \mu P)_{ij} = \lambda_i \delta_{ij}$, where λ_i are the eigenvalues of the μ matrix, and the diagonalizing P may be constructed by using the successive normalized eigenvectors of μ as columns. The appropriate transformation connecting the y-process to the x-process can be simply shown then to be

$$y = \mu P x \tag{9}$$

or alternatively

$$y = P\Lambda x, \tag{10}$$

where y and x are column vectors containing the N values of each sequence, and Λ is the matrix of eigenvalues of μ, such that

$$(\Lambda)_{ij} = \lambda_i \delta_{ij}. \tag{11}$$

Then, with the transformation so constructed, the distribution of the x_i is given to within scale factors by

$$\exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\lambda_i x_i^2\right\}. \tag{12}$$

These x_i are seen to be indeed independent of each other and gaussianly distributed, and hence may be chosen simply as before by choosing a set of random numbers r_i from a random number table, and inverting the relations

$$r_i = \frac{1}{2}\left[1 + \operatorname{erf}\left(x_i\sqrt{\frac{\lambda_i}{2}}\right)\right]. \tag{13}$$

With the x_i thus selected, the corresponding y_i are now obtained by the simple matrix multiplication (9) or (10).

Since the matrix maultiplication is simple, once P and Λ are found, the basic difficulty of the technique proposed has been concentrated on finding the eigenvalues and eigenvectors for the given autocorrelation matrix μ. This may be a formidable computational problem, and is discussed further in a later section.

One interesting point may be noted from (12). Namely, although the y_k are considered as a set of successive points from a stationary gaussian process, the x_k can no longer be so considered, because of the differences in the effective deviation ($\sigma_k = 1\sqrt{\lambda_k}$) associated with each x_k. Another way of looking at this is that the procedure has corrected for the use of only a finite number of terms in the x sequence, by changing the spread associated with some of these. This modification of the uncorrelated process is therefore no longer a simple modification of a white noise process, but actually more subtly uses another process to obtain the desired result. The necessity for doing this to get the desired exact result undoubtedly accounts, at least in part, for the error inherent in the shaping filter concept.

Use of Conditional Probabilities

Still another alternative procedure is based on the use of conditional probabilities, and would be particularly useful if the desired process were governed by a low-order distribution. Thus, if $y_m, \cdots, y_{m+k-1}$ have already been chosen, then y_{m+k} is given in terms of the conditional joint probability by

$$P_{k+1}(y_{m+k} \mid y_m, \cdots, y_{m+k-1}) = \frac{W_{k+1}(y_m, \cdots y_{m+k})}{W_k(y_m, \cdots y_{m+k-1})}. \tag{14}$$

Thus, if $y_m, \cdots, y_{m+k-1}$ have been selected, this gives a probability for y_{m+k} as

$$W(y_{m+k}) = e^{-Ay^2_{m+k}-By_{m+k}} = Ke^{-A(y_{m+k}-\beta)^2}. \tag{15}$$

Thus a value can be chosen for y_{k+m}, consistent with the statistics and the previous k values, by using the same error-function relation as before, with $(y_{m+k} - \beta)$ as the random variable. It should be noted, however, that the evaluation of the coefficients A and B above will require the knowledge of μ^{-1} where μ is both of kth and of $(k+1)$th order. This process can be really convenient then, when the process is assumed governed by a kth order distribution, with k low, where any new point need only be related to $k-1$ previous points. In such a case, the computations would be considerably shortened by using this procedure. Actually, for a kth order governing distribution, the first k points could be obtained simultaneously by the matrix diagonalization described earlier, and succeeding points by the procedure described here. Then only two matrix inversions, of a kth and $(k+1)$th order matrix, would be necessary, because the same matrix coefficients will occur each time. The "low" value mentioned for k need not be severely low, but would be governed probably by the availability of convenient computer programs for handling the matrices involved. In fact the main advantage to this procedure is the fact that if only matrices of a certain low order need be dealt with, it might be possible to make use of computer programs already written out for rapid solution of such matrices. It should however be pointed out that the procedure described above will give exact results, only if the process is actually governed by its kth order probability distribution density.

Discussion of the Autocorrelation Matrix

It is apparent that, in general, diagonalization of the matrix μ would have to be solved by use of a computing machine, and it will be of primary concern as to what order, N, of matrix can be dealt with accurately on a machine. This will be a major problem in obtaining the desired sequences, particularly limiting the length of the sequences. However, it should be emphasized that for any given power spectrum or autocorrelation function, such sequences would have to be computed only once, and would then be available for use in whatever problems the statistics arose.

In addition, it should be added here that it is possible to achieve a completely analytic diagonalization in at least one case of major interest, namely, the gaussian Markov process. Here, assuming $\sigma^2 = 1$, $R(t) = e^{-\alpha|t|}$ and the matrix elements are $\mu_{ij} = e^{-\alpha|(i-j)t|}$ (16) where t is the sampling interval, with $\rho = e^{-\alpha|t|}$. It is readily shown that the eigenvalues are of the form

$$\lambda_i = \frac{1-\rho^2}{1+\rho^2+2\rho\cos\phi_i} \tag{17}$$

where ϕ_i are the successive solutions of the transcendental equation

$$\tan N\phi = \frac{-\sin\phi}{\cos\phi\left(\frac{1+\rho^2}{1-\rho^2}\right)+\frac{2\rho}{1-\rho^2}}. \tag{18}$$

The corresponding result for the eigenvectors is that the kth component of the jth normalized eigenvector is given by

$$x_k^{(j)} = (-)^{N+k}\left(\frac{1}{\lambda_j}\frac{\sin N\phi_j}{\sin \phi_j}\right) \cdot \sqrt{\frac{2}{N(1+\rho^2+2\rho\cos\phi_j)+(1-\rho^2)}} \cdot [\rho \sin (k-1)\phi_j + \sin k\phi_j] \qquad k = 1, \cdots, N. \tag{19}$$

It can easily be shown from the results for λ_j and ϕ_j that

$$\lambda_j = \left|\frac{\sin N\phi_j}{\sin \phi_j}\right| \tag{20}$$

and hence the factor $1/\lambda_j$ $(\sin N\phi_j)/(\sin \phi_j)$ is simply a ± 1 factor, depending on the value of $N\phi_j$.

Thus, in the case of a Markov process the stated problem is completely solved. The precision is limited only to the precision which it is desired to use in solving the transcendental equation for ϕ, and the precision to which subsequent simple arithmetic processes of multiplication and division are carried out.

It is interesting to note the clustering of the eigenvalues about $\lambda = 1$. It is easily verified by checking back, that if $\sigma^2 \neq 1$ is used, the values obtained for the eigenvalues are just multiplied by σ^2, while the eigenvectors remain the same. This clustering is to be expected, since the sum of the eigenvalues is equal to the trace of the autocorrelation matrix. Since all elements of the diagonal are equal to σ^2, this sum is $N\sigma^2$, and hence the average value of the N eigenvalues is always equal to σ^2. This has unfortunate implications for the use of a large scale computer to deal with diagonalization of a more arbitrary autocorrelation matrix, since the order of the matrix which can be handled within a given accuracy may be more severely limited when the eigenvalues are clustered in this fashion, than if they were spread.

Conclusion

The primary purpose of this paper has been to focus attention on a possible method for dealing numerically with a class of problems involving stochastic processes, and to discuss the associated question of how to obtain stochastic samples by computational or analytical means. Should the concepts be of sufficient value, it should be possible to compute out tables of sample sequences once and for all, for use in these procedures.

It has been shown that a sample sequence of N values of a gaussian process can be generated exactly by diagonalization of an Nth order matrix, for example, by machine computation. By use of a general symmetry property for the eigenvectors of such a matrix, it is possible to reduce the order of the matrices to be dealt with by half, although this also loses much of the symmetry of the original matrix. Two possible approximate generating procedures were also suggested, neither of which, however, will give exactly the desired statistics. Although in the latter cases it would be improper to apply the resulting sequences without an analysis of the likely error, the procedures may well be useful for generating long sequences, even though only approximate. Basically, they would obviate the limitations in sequence length caused by the limitation on the order of a matrix which can be handled on a digital computer without large errors.

The other possibility, analytical diagonalization of the Nth order autocorrelation matrix, may be possible in special cases. Results of this type have been given here for the simple gaussian Markov process, which is the only process which has seemed amenable in this manner to date.

Bibliography

[1] Cramer, H., *Methods of Mathematical Statistics*. Princeton, Princeton University Press, 1946, pp. 118–119.
[2] Slepian, D., "Estimation of Signal Parameters in the Presence of Noise," Transactions of the IRE, PGIT-3 (March, 1954), pp. 68–89.
[3] The Rand Corporation, *A Million Random Digits With 100,000 Normal Deviates*. Glencoe, Ill., Free Press, 1955.

A Bibliography of Soviet Literature on Noise, Correlation, and Information Theory*

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Mathematical Characterization of Random Time Functions

Kenderskii, A. M., "On the Distribution of the Absolute Value of the Maximum Deviation from the Mean in a Series of Observations." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 85 (1952), pp. 5–8. [A1][2]

Kunimovich, V. I., and Leontovich, M. A., "On the Distribution of the Number of Large Deviations in Electrical Fluctuations." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 53, No. 1 (1946). In English. [A2]

Kunimovich, V. I., "Fluctuation Processes as Oscillations of Random Amplitude and Phase." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 19 (November, 1949), pp. 1231–1259. [A3][1,5]

Kunimovich, V. I., "Voltage Peaks of Fluctuation Noise; Observed Groups of Peaks." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 21 (1951), pp. 625–646. [A4][1]

Khinchin, A. Ya., "The Theory of Correlation of Stationary Chance Processes," *Uspekhi Matematicheskikh Nauk* (*Progress in the Mathematical Sciences*), No. 5 (1938). [A5]

Khusu, A. P., "On Certain Funcationals, Encountered in Engineering, Given by Processes." *Vestnik Leningrad University*, Mathematical, Mechanical, Astronomical Series No. 1 (1956). [A6]

Kolmogorov, A. N., *The Statistical Theory of Oscillations Having a Continuous Spectrum.* Jubilee Collection, AN USSR, Part 1, 1947. [A7]

Kuznetsov, P. I., Stratonovich, R. L., and Tikhonov, V. L., "On the Duration of Excursions of a Random Function." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 24 (1954), pp. 103–112. [A8][5]

Kuznetsov, P. I., Stratonovich, R. L., and Tikhonov, V. L., "Correlation Functions in the Theory of the Brownian Movements." (Generalization of the Fokker-Planck Equation.) *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 24, No. 2 (1954). [A9][7]

Kuznetsov, P. I., Stratonovich, R. L., and Tikhonov, V. L., "Quasi-Moment Functions in the Theory of Random Processes." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 94 (1954), pp. 615–618. [A10][2]

Pugachev, V. S., "General Theory of the Correlation of Stochastic Functions." *Izvestia Akademiia Nauk SSSR, Seria matematicheskaia* (*Bulletin, Academy of Sciences of the USSR, Mathematical Series*), Vol. 17, No. 5 (October, 1953), pp. 401–420. [A11][2]

Rytov, S. M., "Relation of the Distribution of a Quasi-Monochromatic Stationary Process to the Distribution of its Envelope." *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* (*Journal of Experimental and Theoretical Physics*), Vol. 29, No. 5 (11) (1955), pp. 702–703. [A12][5]

Tikhonov, V. I., "Distribution of Excursions of Normal Fluctuations According to Duration." *Radiotekhnika i Elecktronika*, Vol. 1, No. 1 (January, 1956), pp. 23–33. [A13][5]

Yaglom, A. M., "On the Statistical Treatment of Brownian Motion." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 56 (1947), p. 691. [A14][2]

Yaglom, A. M., "Introduction to Theory of Stationary Random Processes." *Uspekhi Matematicheskikh Nauk* (*Progress in the Mathematical Sciences*), Vol. 7 (September, 1952), pp. 3–168. [A15][2]

Yaglom, A. M., "Correlation Theory of Processes with Random Stationary *n*-th Increments." *Matematicheskii Sbornik* (*Mathematical Review*), Vol. 37 (79), 1, (1955), pp. 141–196. [A16][5]

Yaglom, A. M., and Pinsker, M. S., "Random Processes with Fixed Increment of the *n*th Order." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 90, No. 5 (June, 1953), pp. 731–734. [A17]

Physical Origins of Noise

Gertsenshtein, M. E., "Theory of Shot Effect; Correlation of Fluctuations in Electron Gases." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 25, No. 5 (May, 1955), pp. 827–840. [B1]

Ginzburg, V. L., "Certain Questions of the Theory of Electric Fluctuations." *Uspekhi Fizicheskikh Nauk* (*Progress in the Physical Sciences*), Vol. 46, No. 3 (1952), pp. 348–387. [B2]

Ginzburg, V. L., "Again on the Application of Thermodynamics to Electric Fluctuations." *Uspekhi Fizicheskikh Nauk* (*Progress in the Physical Sciences*), Vol. 52, No. 3 (March, 1954), pp. 494–497. [B3]

Gorelik, G. S., "Certain Applications of the Second Law of Thermodynamics to Electrical Fluctuations." *Uspekhi Fizicheskikh Nauk* (*Progress in the Physical Sciences*), Vol. 44, No. 1 (1951), pp. 33–45. [B4][1,3]

Leontovich, M. A., and Rytov, S. M., "Differential Law for the Intensities of Electrical Fluctuations and Influence of Skin Effect." *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* (*Journal of Experimental and Theoretical Physics*), Vol. 23 (1952), pp. 246–252. [B5][1]

Levin, M. L., "Thermodynamic Examination of Balanced Electric Fluctuations." *Uspekhi Fizicheskikh Nauk* (*Progress in the Physical Sciences*), Vol. 52, No. 3 (March, 1954), pp. 486–493. [B6]

Levin, M. L., "Induced Thermal Noise in Antennas." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 25, No. 13 (1955). [B7]

Levin, M. L., and Ginzburg, V. L., "Application of Thermodynamics to Electric Fluctuations." *Uspekhi Fizicheskikh Nauk* (*Progress in the Physical Sciences*), Vol. 56, No. 1 (April, 1955). [B8]

Levin, M. L., and Rytov, S. M., "Thermal Radiation of Thin Rectilinear Antennas." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 25, No. 2 (February, 1955), pp. 323–332. [B9]

Petrova, L. Ya., "Shot Noise in Semiconductors." *Radiotekhnika i Elektronika*, Vol. 1, No. 1 (January, 1956), pp. 98–106. [B10]

Pumper, Ye. Ya., "Fluctuations of Electric Current in Electronic Tubes with Tungsten Cathode." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 59, No. 9 (January, 1948). [B11][1]

Rytov, S. M., "Electric Fluctuations and Thermal Radiations." *Uspekhi Fizicheskikh Nauk* (*Progress in the Physical Sciences*), Vol. 55, No. 3 (March, 1955), pp. 299–314. [B12]

Rytov, S. M., "Theory of Thermal Noise." (Parts I and II.) *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 2 (1955), pp. 3–13 and Vol. 10, No. 3 (1955), pp. 3–13. [B13][5]

Troitskii, V. S., Lyubina, A. G., and Zolotov, A. V., "Experimental Study of the Thermal Noise of Some Conductors." *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* (*Journal of Experimental and Theoretical Physics*), Vol. 25, No. 4 (10) (1953), pp. 455–462. [B14][1]

Vasil'yev, Ye. I., and Lopukhin, V. M., "Theory of Noises in Traveling-Wave Tubes." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 24 (May, 1954), pp. 895–898. [B15][1]

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[1] An abstract or review is available in English as translation number RT-3792 from the Scientific Translations Center, Science Division, Library of Congress, Washington, D.C.

[2] An abstract or review has been published in the American Mathematical Society's *Mathematical Reviews* (usually within 12 months of the appearance of the article).

[3] A full translation is available from the same source as in footnote 1.

[4] A full translation is available from Armed Services Technical Information Agency, Document Service Center, Knott Bldg., Dayton 2, Ohio.

[5] A full translation is available from Morris D. Friedman, 572 California St., Newtonville 60, Mass.

[6] A full translation has appeared in the American Physical Society's *Soviet Physics JETP*.

[7] A full translation is available from Columbia Technical Translations, 561 W. 141st St., New York, N.Y.

[8] A translated table of contents is available from the same source as in footnote 1.

Noise in Linear Systems

Dementeev, E. P., "The Analysis of the Noise Properties of UHF Amplifiers." *Radiotekhnika* (*Radio Engineering*), Vol. 6, No. 2 (1951), pp. 10–18. [C1]

Dementeev, E. P., "Noise of Super-High Frequency Amplifiers." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 1 (1955), pp. 37–44. [C2][5]

Gal'perin, B. S., "Contact Noises." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 25 (March, 1955), pp. 410–412. [C3][1,5]

Kolosov, A. A., "Noise Bands of Multistage Resonance Amplifiers." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 57, No. 4 (1948). [C4][1]

Kuznetsov, P. I., Stratonovich, R. L., and Tikhonov, V. L., "Passage of Certain Random Functions Through Linear Systems." *Avtomatika i telemekhanika* (*Automatics and Telemechanics*), Vol. 14 (1953), pp. 144–163. [C5][1]

Levin, M. L., "Influence of Chance Fluctuations of the Parameters of Long Lines on the Propagation of Harmonic Signals." *Isvestiia Akademiia Nauk SSSR, Otdelenie tekhnicheskikh nauk* (*Bulletin of the Academy of Sciences of the USSR, Department of Technical Sciences*), No. 9 (September, 1953), pp. 1366–1367. [C6]

Siforov, V. I., "An Investigation of Amplifier Noise at Ultra High Frequencies." *Radiotekhnika* (*Radio Engineering*), Vol. 3, No. 3 (May-June, 1948). [C7][1]

Troitskii, V. S., "Fluctuations in Loaded Lines." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 25, No. 8 (1955), pp. 1426–1435. [C8][5]

Noise in Nonlinear Systems

Amiantov, I. N., and Tikhonov, V. I., "The Effect of Normal Fluctuations on Typical Nonlinear Elements." *Izvestiia Akademi'a Nauk SSSR, Otdelenie tekhnicheskikh nauk* (*Bulletin of the Academy of Sciences of the USSR, Department of Technical Sciences*), No. 4 (1956), pp. 33–42. [D1]

Bershtein, I. L., "Detection of a Continuous Spectrum." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 11 (1941), p. 30. [D2][1]

Bunimovich, V. I., "Effect of Fluctuation and Signal Voltages on a Nonlinear System." *Zhurnal Fiziki* (*Journal of Physics, USSR*), Vol. 10, No. 1 (1946), pp. 35–48. In English. [D3][1]

Bunimovich, V. I., "Conversion of Fluctuations with Nonlear Systems." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 16, No. 6 (1946). [D4]

Bunimovich, V. I., "On the Passage of Signal and Noise Through a Limiter and Differentiator." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 6 (1955), pp. 3–16. [D5][1,5]

Gutkin, L. S., "Interaction of Signal and Noise in an Inertial Detector." In two parts. *Radiotekhnika* (*Radio Engineering*), Vol. 11, No. 2 (1956), pp. 42–53 and Vol. 11, No. 3 (1956), pp. 51–62. [D6]

Kuznetsov, P. I., Stratonovich, R. L., and Tikhonov, V. I., "Propagation of Random Functions Through Nonlinear Systems." *Avtomatika i telemekhanika* (*Automatics and Telemechanics*), Vol. 15 (1954), pp. 200–205. [D7][1]

Malakov, A. N., "Stabilization of Random Voltage Fluctuations by a Ferroresonant Stabilizer." *Izvestiia Akademiia Nauk SSSR, Otdelenie tekhnicheskikh nauk* (*Bulletin of the Academy of Sciences of the USSR, Department of Technical Sciences*), No. 7 (1955), pp. 3–9. [D8][1]

Manaev, E. I., "The Structure of Smooth Interference, and the Transmission of Signal and Interference Through a Linear Inertialess Detector." *Radiotekhnika* (*Radio Engineering*), Vol. 6, No. 2 (1951), pp. 19–31. [D9]

Manaev, E. I., "Simultaneous Detection of Signal and Interference by Amplitude- and Phase-Detectors Operating in a Linear Non-Intertial Mode." *Radiotekhnika* (*Radio Engineering*), Vol. 9 (1954), pp. 19–32. [D10][3]

Pestryakov, V. B., "The Effect of Periodic Interference on the Reception of FM Signals When a Current Limiter is Used and the N/S Ratio is High." *Izv. Elek. Slab. Toka*, Vol. 7 (1938), p. 26. [D11]

Rimskii-Korsakov, A. V., "Nonlinear Distortion of Nonperiodic Signals in an Electroacoustic Canal." *Akusticheskii Zhurnal* (*Journal of Acoustics*), Vol. 1, No. 2 (1955), pp. 165–170. [D12][1,5]

Rimskii-Korsakov, A. V., "Computation of the Audibility of Nonlinear Distortion Which Arises in an Electro-Acoustic Channel." *Akusticheskii Zhurnal* (*Journal of Acoustics*), Vol. 2, No. 1 (1956). [D13][1]

Rosov, M. W., "Excursions after a Quadratic Detector." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), No. 7 (July, 1952), pp. 1195–1200. [D14][5]

Tikhonov, V. I., "Effect of Rectangular Pulses with Random Duration and Interval on a Linear Detector." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 12 (1955). [D15][1]

Vol'f, V. M., "Investigation of Nonlinear Distortions of Quadrupoles by means of Continuous-Spectrum Signals." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 24, No. 11 (November, 1954), pp. 2054–2063. [D16]

Noise in Oscillating Systems

Bershtein, I. L., "On Fluctuations in the Neighborhood of Periodic Motion of an Auto-Oscillating System." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 20 (1938), p. 11. In English. [E1]

Bershtein, I. L., "Fluctuations in Auto-Oscillating Systems and the Determination of the Frequency Deviations of a Valve Oscillator." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 11, No. 4 (1941), p. 305. [E2][1]

Bershtein, I. L., "The Fluctuations in Amplitude and Phase in a Valve Oscillator." *Izvestiia Akademiia Nauk SSSR, Seriia fizicheskaia* (*Bulletin of the Academy of Sciences of the USSR, Physical Series*), Vol. 14, No. 2 (1950), pp. 145–173. [E3][1]

Bershtein, I. L., "Fluctuations of a Klystron Generator." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 106, No. 3 (1956). [E4]

Gonorovskii, I. S., "Fluctuation Characteristics of Oscillation Formation in an Electronic Autogenerator." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 94, No. 5 (February, 1954), pp. 869–872. [E5]

Gonorovskii, I. S., "On Phase Fluctuations in a Tube Oscillator." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 101, No. 4 (1955), pp. 657–660. [E6][5]

Gorelik, G. S., "Problem of the Technical and Natural Width of Line in a Tube Oscillator." *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* (*Journal of Experimental and Theoretical Physics*), Vol. 20, No. 4 (1947). [E7][1]

Gorelik, G. S. "Non-Linear Oscillations, Interference, and Fluctuations." *Izvestiia Akademiia Nauk SSSR, Seriia fizicheskaia* (*Bulletin of the Academy of Sciences of the USSR, Physical Series*), Vol. 14, No. 2 (1950). [E8][1]

Kuznetsov, P. I., Stratonovich, R. L., and Tikhonov, V. I., "On the Action of Electric Fluctuations on a Tube Oscillator." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 97, No. 4 (1954).[5] Also *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* (*Journal of Experimental and Theoretical Physics*), Vol. 28, No. 5 (May, 1955), pp. 509–523.[6] [E9][1]

Rytov, S. M., "Fluctuations in Oscillating Systems of the Thompson Type, Parts I and II." *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* (*Journal of Experimental and Theoretical Physics*), Vol. 3, No. 9 (1955). [E10][1]

Rytov, S. M., "On the Question of Fluctuating Phase in a Tube Generator." *Radiotekhnika i Elektronika*, Vol. 1, No. 1 (January, 1956), pp. 114–120. [E11]

Zhabotinskii, M. E., "On Fluctuations in an Oscillator with an Inertial Nonlinearity." *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* (*Journal of Experimental and Theoretical Physics*), Vol. 26, No. 6 (1954), pp. 758–759. [E12][5]

Mean-Square Error Minimization

Kolmogorov, A. N., "Interpolation and Extrapolation of Stationary Chance Series." *Izvestiia Akademiia Nauk SSSR, Seria matematicheskaia* (*Bulletin, Academy of Sciences of the USSR, Mathematical Series*), No. 5 (1941). [F1]

Krein, M. G., "Fundamental Approximation Problem in the Theory of Extrapolation and Filtering of Stationary Random Processes." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 94, No. 1 (January, 1954), pp. 13–16. [F2][2]

Kurakin, K. I., "On the Question of the Quality of a Servo and an Automatic Regulation System." *Avtomatika i telemekhanika* (*Automatic and Telemechanics*), Vol. 12, No. 2 (1951). [F3]

Kurakin, K. I., "On the Choice of Optimum Characteristics of a Linear Servo." *Avtomatika i telemekhanika* (*Automatics and Telemechanics*), Vol. 14, No. 4 (1953). [F4]

Kurakin, K. I., "On the Choice of the Optimum Characteristics of a Linear Differentiator in Automatic Control Systems." *Avtomatika i telemekhanika* (*Automatics and Telemechanics*), Vol. 16, No. 3 (1955), pp. 293–299. [F5][1,5]

Leonov, Ya. P., "On the Question of Filtration of Nonstationary Random Functions." *Avtomatika i telemekhanika* (*Automatics and Telemechanics*), Vol. 17, No. 2 (February, 1956), p. 97. [F6][3]

Pinsker, M. S., and Yaglom, A. M., "Linear Extrapolation of Random Processes with Stationary n-tuple Increments." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 94, No. 3 (January, 1954), pp. 385–388. [F7][2]

Pugachev, V. S., "General Theory of Random Functions and its Application to Automatic Regulation." *Proceedings of Second All-Union Conference on Auto. Reg.*, Vol. 2 (November, 1953), pp. 403–425. [F8][1]

Shtein, V. M., "On the Computation of Linear Predicting and Compensating Devices." *Radiotekhnika* (*Radio Engineering*), Vol. 11, No. 2 (1956), pp. 60–63. [F9]

Principles of Automatic Regulations, V. V. Solodovnikov, ed. 1955, 1150 pp., Chapters 22 and 23. [F10]

Solodovnikov, V. V., "Analysis and Synthesis of Tracking and Control Systems under the Influence of Fixed Random Effect." *Izvestiia Akademiia Nauk SSSR, Otdelenie tekhnicheskikh nauk* (*Bulletin of the Academy of Sciences of the USSR, Department of Technical Sciences*), No. 11 (1950), pp. 1648–1679. [F11][1]

Solodovnikov, V. V., "On the Dynamical Exactness and Optimal Characteristics of Follower Systems and Transforming Devices." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 77, No. 2 (March, 1951). [F12]

Solodovnikov, V. V., and Matveev, P. S., "Synthesis of Compensators for Following Systems when Interference is a Given Requirement for Dynamic Accuracy." *Avtomatika i telemekhanika* (*Automatics and Telemechanics*), Vol. 16, No. 3 (1955), pp. 233–257. [F13][1,5]

Yaglom, A. M., "On Problems of Linear Interpolation of Stationary Random Sequences and Processes." *Uspekhi Matematicheskikh Nauk* (*Progress in the Mathematical Sciences*), Vol. 4, No. 4 (32) (1949), pp. 173–178. [F14]

Yaglom, A. M., "Effective Solutions of Linear Approximation Problems for Processes with Random Stationary *n*th Increments." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 98, No. 2 (September, 1954), pp. 189-192. [F15][2,5]

Yaglom, A. M., "Extrapolation, Interpolation, and Filtration of Stationary Random Processes with Rational Spectral Density." *Trudy, Moscow Mathematical Society*, Vol. 4 (1955), pp. 333-375. [F16]

Statistical Communication Theory

Blokh, E. L., and Kharkevich, A. A., "Geometric Theory of the Capacity of a Communication System." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 7 (1955), pp. 3–7. [G1][1,5]

Blokh, E. L., and Kharkevich, A. A., "Geometric Representations in Communication Theory." *Izvestiia Akademiia Nauk SSSR, Otdelenie tekhnicheskikh nauk* (*Bulletin of the Academy of Sciences of the USSR, Department of Technical Sciences*), No. 6 (1955), pp. 91–100. [G2]

Fink, L. M., "Apropos 'The Limiting Capacity of a Communication System' by A. A. Kharkevich and E. L. Blokh." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 10 (1955), pp. 74–75. [G3][5]

Kharkevich, A. A., "Basic Features of a General Theory of Communication." *Radiotekhnika* (*Radio Engineering*), Vol. 9, No. 5 (1954). [G4][3]

Kharkevich, A. A., and Blokh, E. L., "On the Limiting Capacity of a Communication System." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 2 (1955), pp. 14–20. [G5][1,5]

Khinchin, A. Ya., "The Concept of Entropy in the Theory of Probability." *Uspekhi Matematicheskikh Nauk* (*Progress in the Mathematical Sciences*), Vol. 8, No. 3, (55) (1953), pp. 3–20. [G6][2,3]

Khinchin, A. Ya., "On the Principal Theorems of Information Theory." *Uspekhi Matematicheskikh Nauk* (*Progress in the Mathematical Sciences*), Vol. 11, No. 1 (67) (1956), pp. 17–74. [G7][5]

Kolmogorov, A. N., *Concept of Information, Methods of Measuring the Amounts of Information, and Principles of the Shannon Theory of Information Transmission.* Talk at Conference on the Theory of Probability and Its Applications, Leningrad, May 30–June 4, 1955. [G8]

Kotel'nikov, V. A., "The Interference Rejection of Different Types of Lines with Regard to Fluctuating Interference." *Radiotekhnika* (*Radio Engineering*), No. 2 (May, 1946), pp. 75–83. [G9]

Kotel'nikov, V. A., *Theory of Potential Noise Stability.* Doctorate dissertation, Moscow Energetics Institute, 1946. (G10]

Kotel'nikov, V. A., "The Potential Interference Rejection in Fluctuating Interferences." *Radiotekhnika* (*Radio Engineering*), No. 6 (June, 1947), pp. 55–66. [G11]

Kotel'nikov, V. A., "Problems of Interference Immunity in Radio Communication." *Radiotekh. Sbornik* (1947), pp. 5–12. [G12][3]

Losyakov, B. N., "Fundamental Relationship Between Signal Strength, Bandwidth, and Signal to Noise Ratio in Radio Communication." *Radiotekhnika* (*Radio Engineering*), Vol. 5, No. 1 (1950), pp. 28–41, and No. 4 (July-August, 1949), pp. 53–61. [G13][1]

Nazarov, M. V., *Theory of Signal Separation.* Cand. dissertation, Faculty of Radio Communication and Radio Broadcasting, Moscow Electrotechnical Institute of Communications, 1953–1954. [G14]

Neyman, M. S., "General Theory of Signals and General Theory of Automatic Processes." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 5 (1955). [G15][1,3]

Pinsker, M. G., "Amount of Information of a Gaussian Process with Respect to a Gaussian Process Relating to it in a Stationary Way." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 99 (1954), pp. 213–216. [G16][1]

Vasil'yev, Ye. I., "Geometric Derivation of the Channel Capacity of a System Using a Special Type Receiver." *Radiotekhnika* (*Radio Engineering*), Vol. 11, No. 2 (1956). [G17][5]

Yushkevich, A. A., "Limit Theorems Connected with the Concept of Entropy of Markov Chains." *Uspekhi Matematicheskikh Nauk* (*Progress in the Mathematical Sciences*), Vol. 8, No. 5 (September-October, 1953), pp. 177–180. [G18][1]

Ziuko, A. G., *Capacity of Radio Lines and Receiver Interference-Stability at High Fluctuation Interference Levels.* Cand. dissertation, Moscow Electrotechnical Institute of Communications, 1952. [G19]

Ziuko, A. G., "Comparative Estimate of Communication Channels with Different Modulation Systems According to Their Capacity." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 6 (1955), pp. 58–66. [G20][5]

Effect of Random Disturbances on Communication Systems

Amarantov, V. N., *Distortions of the Signal in a Tone Telegraphy Channel by Harmonic Noise, Fluctuation Noise, and Level Changes.* Cand. dissertation, Technical Science, Leningrad Institute of Communication Engineers, 1954. [H1]

Borodich, S. V., "Noise Stability of Communication with Pulse-Code Modulation." *Radiotekhnika* (*Radio Engineering*), Vol. 4, No. 5 (September-October, 1949), [H2][3]

Bykov, Yu. S., "The Interference Rejection of Radiotelephone Communication Systems." *Radiotekhnika* (*Radio Engineering*), No. 4 (July-August, 1948), pp. 59–68. [H3][1]

Filippov, L. I., "On the Noise Stability of Impulse Radio Reception." *Radiotekhnika* (*Radio Engineering*), No. 6 (1954). [H4]

Filippov, L. I., "Potential Interference-Stability in the Reception of Pulse Radio Signals." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 10 (1955). [H5][5]

Finkel'shteyn, M. I., "The Increase of Signal-to-Noise Ratio for Receivers of Pulse Radio Signals." *Radiotekhnika* (*Radio Engineering*), Vol. 5, No. 5 (1950), pp. 68–76. [H6]

Gutkin, L. S., "Effect of Interference on Super-Regenerative Reception." (In three parts.) *Radiotekhnika* (*Radio Engineering*), Vol. 1, No. 9 (1946), Vol. 2, No. 4 (1947), and Vol. 4, No. 1 (1949). [H7][1]

Ivanov, P. N., "The Effective Sensitivity of Receivers Operating under Conditions of Pulse Interference." *Radiotekhnika* (*Radio Engineering*), No. 1 (January-February, 1948), pp. 80–81. [H8]

Izyumov, N. M., "The Required Signal Strength at the Receiver Input in Pulse-Phase Radio Communications." *Radiotekhnika* (*Radio Engineering*), No. 4 (July-August, 1948), pp. 59–68. [H9][1]

Lebedev, V. L., "Several Questions of Impulse and Fluctuation Influence on Radio Receiving Apparatus." Cand. dissertation, Technical Science, Moscow Power Engineering Institute, 1953. [H10]

Levin, G. A., and Govolchiner, M. M., "Analysis of Quantization Noise in Pulse-Code Modulation." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 8 (1955), pp. 3–21. [H11]

L'vovich, A. A., "Dynamic Sensitivity and Interference Rejection in Radiotelegraph Equipment." *Radiotekhnika* (*Radio Engineering*), No. 4 (July-August, 1948), pp. 59–68. [H12][1]

L'vovich, A. A., "Methods of Calculating the Signal/Noise Ratio and Sensitivity of Radio Telegraphy Receivers." *Radiotekhnika* (*Radio Engineering*), Vol. 4 (May-June, 1949), pp. 57–68. [H13][3]

Mel'nikov, V. S., "Measuring Sensitivity of Receivers with the Aid of a Noise Diode Tube." *Vest. Svyazi*, No. 10 (October, 1948). [H14][1]

Shastova, G. A., "Investigation of the Noise Stability in the Transmission of Command Signals in Telecontrol Systems by Means of the Theory of Potential (Limit) Stability Against Noise, Part I." *Avtomatika i telemekhanika* (*Automatics and Telemechanics*), No. 4 (July-August, 1955). [H15][1,5]

Siforov, V. I., "The Effect of Interference on Radio Reception of Pulse Signals." *Radiotekhnika* (*Radio Engineering*), Vol. 1, No. 7 (1946). [H16][1]

Siforov, V. I., "Analysis of the Passage of Signal Impulses and Noise Through the Receiver of a Pulse-Radio-Navigation Device." *Trudy, Leningrad Red Banner Military-Preparatory Engineering Academy*, No. 13 (1947). [H17]

Siforov, V. I., "Irregularities in Radio Engineering." *Radiotekhnika* (*Radio Engineering*), No. 11 (1953), pp. 6–8. [H18]
Vollerner, N. F., "Comparative Noise Stability of Radio Communication with AM, FM, and PM." *Radiotekhnika* (*Radio Engineering*), Vol. 5, No. 1 (1950), pp. 21–27. [H19]
Zhitomirskii, V. I., "Determination of the Probability of Communication Interference Caused by Interfering Signals." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 10 (1955). [H20][1,5]
Zingerenko, A. M., "Distortion of the Length of Telegraph Impulses under the Action of Fluctuation Noise in an f.m. System." *Radiotekhnika* (*Radio Engineering*), Vol. 11, No. 3 (March, 1956), pp. 70–74. [H21]

Weak-Signal Detection and Suppression of Interference

Al'pert, Ia. L., and Rytov, S. M., "Suppression of Radio Noise." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 14, No. 12 (1944), pp. 730–748. [I1][1,3]
Balubka, D. K., Miashikov, L. L., and Plotnikov, E. N., *Modulation Method of Measuring Low Pressures in the A. F. Region.* Conference on Physical Acoustics and Ultrasonics, Moscow, March 3–7, 1955. [I2][1]
Basharinov, A. E., "On the Noise Stability of the Correlation Method of Reception." *Radiotekhnika* (*Radio Engineering*), Vol. 11, No. 5 (1956), pp. 26–34. [I3]
Belousov, A. P., "Increasing the Effective Sensitivity of a Pulse Receiver." *Radiotekhnika* (*Radio Engineering*), No. 6 (June, 1947), pp. 55–66. [I4]
Bunimovich, V. I., "Sensitivity of a Radiometer." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 20 (August, 1950), pp. 944–953. [I5][1,3]
Bunkin, F. V., and Karlov, N. V., "On the Sensitivity of Radiometers." (Parts I and II.) *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 25, No. 3 (1955), pp. 430–435, and Vol. 25, No. 4 (1955), pp. 733–741. [I6][5]
Bykov, Yu. S., "The Design of Communications Channels for Use in Conditions of Intense Noise." *Radiotekhnika* (*Radio Engineering*), No. 2 (May, 1946), pp. 75–83. [I7]
Bystrov, G. N., "On the Question of the Dependence Between the Observability of an Object and the Number of Illuminating Pulses." *Radiotekhnika* (*Radio Engineering*), Vol. 11, No. 2 (1956), pp. 74–76. [I8][5]
Chaikovskii, V. I., "Reception of Pulse Signals by the Mutual Correlation Method." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 6, (1955), pp. 16–20. [I9][5]
Chaikovskii, V. I., "Noise Stability of a Filtering Autocorrelation Receiver of Signal Impulses." *Radiotekhnika* (*Radio Engineering*), Vol. 11, No. 4 (1956), pp. 24–30. [I10]
Filippov, L. I., "Interference Reduction of Impulse Radio Reception." *Radiotekhnika* (*Radio Engineering*), Vol. 9, No. 6 (1954), pp. 40–48. [I11]
Gorelik, G. S., "Employment of the Modulation Method in Optical Interferometry." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 83, No. 4 (1952), pp. 549–552. [I12][1]
Karlov, N. Y., and Salomonovich, A. E., "The Automatic Null Radiometer of Centimeter Range for Studying Weak Noise Signals." *Radiotekhnika i Elektronika*, Vol. 1, No. 1 (January, 1956), p. 121. [I13]
Karnovskii, M. I., "Suppression of Fluctuation Interference in the Correlation Method of Reception." *Radiotekhnika* (*Radio Engineering*), Vol. 9, No. 3 (1954), pp. 3–11. [I14][5]
Kharkevich, A. A., "The Detection of Weak Signals." *Radiotekhnika* (*Radio Engineering*), Vol. 8, No. 5 (1953). See also "About the Article of A. A. Kharkevich" by A. A. Kharkevich in answer to questions raised by A. K Rosov, *Radiotekhnika* (*Radio Engineering*), Vol. 9, No. 2 (March-April, 1954), p. 77. [I15]
Klyaznik, V. A., *Increasing the Interference Rejection of Radio Reception by Means of Compensation for the Harmful Action of Pulse Interference in the Receiver.* Cand. dissertation, Mathematical Science, 1953 Moscow Electrical Engineering Institute of Communications. [I16]
Neyman, S. A., and Petrovich, N. T., *Radio Interference and Methods of Increasing the Noise Stability of Radio Lines.* Referatnii Sb., Min. Promyshlennosti Sredstv Svyazi, SSSR, Otdel Naukhno-Tekh. Info., 1949. [I17]
Shchukin, A. N., "Application of the Theory of Probability to the Effect of Fading." *Tekh. Fiz. USSR* (1937). In English. [I18]
Shchukin, A. N., "A Method of Noise Pulse Suppression in Radio Reception." *Izvestiia Akademiia Nauk SSSR, Seriia fizicheskaia* (*Bulletin of the Academy of Sciences of the USSR, Physical Series*), Vol. 10, No. 1 (1946), pp. 49–56. [I19][1]
Troitskii, V. S., "On the Theory of Measuring Weak Signals Having a Complex Spectrum." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 21, No. 8 (1951), pp. 994–1003. [I20][4]
Troitskii, V. S., "The Null Method of Measuring Weak Electrical Fluctuations." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 25, No. 3 (1955), pp. 478–496. [I21][5]
Troitskii, V. S., "The Sensitivity of Radiometers." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 22 (March, 1952), pp. 455–460. [I22]
Vitkevitch, V. V., "Wideband Radio Interferometer." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 91, No. 6 (1953), pp. 1301–1303. [I23][5]
Vladimirskii, K. V., "On Synchronous Filters." *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* (*Journal of Experimental and Theoretical Physics*), Vol. 21, No. 1 (1951), pp. 3–10. [I24][3]

Miscellaneous

Aleksandrov, G. A., *Methods for Measuring Fluctuating Noise.* Cand. dissertation, Mathematical Science, Moscow Electrical Engineering Institute of Communications, April 8, 1954. [J1]
Belkin, B. G., "New Noise Voltage Generator." *Radiotekhnika* (*Radio Engineering*), Vol. 10, No. 4 (1955), pp. 56–58. Also, Kiev Conference on Electroacoustics, July 1–5, 1955. [J2][1]
Gershman, S. G., "Correlation Coefficient as a Criterion of Acoustic Quality of a Closed Room." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*) (1951), pp. 1492–1596. [J3][1,3]
Gershman, S. G., "Interference Method of Measuring Correlation Coefficients of Stationary Noises." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus, Academy of Sciences of the USSR*), Vol. 92, No. 1 (September, 1953), p. 33. Also, *Trudy Kom. po Akust.*, Vol. 8 (1955), pp. 151–159. [J4][3]
Gershman, S. G., and Feinberg, E. L., "On the Measurement of the Correlation Coefficient." *Akusticheskii Zhurnal* (*Journal of Acoustics*), Vol. 1, No. 4 (1955), pp. 326–338. [J5][1,5]
Karnovskii, M. I., "Comparison of the Suitability of White Noise, Multitone, and Wailing Sounds for Indoor Acoustical Measurements." *Trudy Kom. po Akust.*, Vol. 8 (1955), pp. 160–167. [J6]
Pumper, Ye. Ya., "A Method for Experimental Investigation of the Statistics of Electrical Fluctuations." *Doklady Akademiia Nauk SSSR* (*Comptes Rendus Academy of Sciences of the USSR*), Vol., 53 (1946), in English; also Vol. 53, No. 1 (July 10, 1946), pp. 25–27. [J7]
Tikhonov, V. I., "Measurement of Electric Fluctuations by Means of Thermoelectric Instruments." *Zhurnal Tekhnicheskoi Fiziki* (*Journal of Technical Physics*), Vol. 25, No. 5 (May, 1955), pp. 817–822. [J8]

Books

Bunimovich, V. I., *Fluctuation Processes in Radio Receivers.* Sovietskoe Radio, 1951, 355 pp. Same as dissertation for Doctor of Physico-Mathematical Science, Institute of Radio Engineering and Electronics, AN USSR, November 25, 1954. [K1][8]
Dolukhanov, M. P., *Introduction to the Theory of Information Transmission Over Electric Communication Channels.* Moscow, Sviaz'izdat, 1955, 126 pp. [K2]
Granovskii, V. L., *Electrical Fluctuations.* Moscow-Leningrad, 1936. [K3]
Kharkevich, A. A., *A Survey of General Communication Theory.* Moscow, Gostekhteorizdat, 1955, 268 pp. [K4][8]
Leontovich, M. A., *Statistical Physics.* Moscow-Leningrad, 1944. [K5]
Rytov, S. M., *Theory of Electric Fluctuation and Thermal Radiation.* Moscow, Ak. Nauk Press, 1953. [K6]
Solodovnikov, V. V., *Introduction to the Statistical Dynamics of Automatic Control Systems.* Moscow-Leningrad, Goztekhteorizdat, 1952. [K7][4]
Theory of Transmission of Electric Signals in the Presence of Noise, N. A. Zheleznov, ed. Translations of papers by Shannon, Rice, Woodward, and Davies. Moscow, Foreign Lit. Publishing House, 1953, 284 pp. [K8]

Abstract

On the Information Invariant*

SATIO OKADA†

The contents of the book "Cybernetics," by N. Wiener, is far more inclusive than conventional electrical engineering or mathematics. As the title of this paper suggests, its intention is not to treat a part of so-called "cybernetics" but to specify a mathematical invariant of information, which will enable us to treat all problems of the field from a unified but nonstatistical standpoint.

As the first step, an ordinary definition of information quantity and its application are given in the first part. This is a brief introduction of a previous paper by the author.[1]

At first, the essential difference between transmissions of energy and information is explained by an example of "common battery" telephone system. Then the quantity of information is defined by quantization of continuous change from simplest telegraph to solid color television. And in all cases, there holds

$$\log C = \frac{1}{\kappa} FT \log e_i$$

where

C = number of permutation,
κ = factor depending on transient, discriminating ability, accuracy of intensity of signal,
F = frequency band,
T = communicating time,
e_i = number of kinds of intensity.

Physical and information theoretic meaning of κ is discussed in detail, especially in reference to absolute delay of transmission and necessity of history caused by reduction of κ.

As its applications, the following are given:

1) Importance of time-division multiplex telephone system.
2) Principle of secret telephone by diluting with camouflage information.
3) Practical limit of compander and expander by quantization principle.
4) Axiomatic possibility of substantial compression of band by a model in thought based on acoustic spectrum by Roland concave grating.
5) High information capacity of high-frequency phenomena (not only electromagnetic but also gravitational, if possible).
6) Possibility of filtering and separating of electric waved by prism and grating.
7) Importance of flip-flop (Kallitrotron) as high information element.
8) Conversion between audition and vision, etc.

However, if we give a glance at any communication method, we find that most communicating apparatuses were invented by appropriate reduction of original information quantity. It means that *adequate dispatching* in information quantity *enables us to invent a new communication device.* Also, the information quantity of the silent worship of Catholics, Quakers, or "Zazen" of Japanese Buddhism, is a little difficult to determine. Then what is kept invariant during these information reducing processes? The previous example represents extremely poor information from the standpoint of the present information theory.

Proposals and suggestions as to an answer of this problem form the second part. At first, "contents" of two informations are defined to be equivalent, if both cause identical phenomenon which is independent of energy consumed for these informations. It can satisfy three axioms of equivalence: Reflexive, symmetric, and transitive. Also "inclusion" of lattice theory is defined by caused phenomena or behaviors. When lattice theory is applied, atoms of elements in "Hasse diagram" may be regarded as absolute information units. The adjective "semantic" in the information theory seems to be an investigation of logical construction of contents, but it can be ultimately regarded as correspondence with each part of behaviors or phenomena.

* *Bull. of Yamagata University*, Eng., Vol. 3, pp. 95–111; May, 1954. Copies are obtainable directly from the author.

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[1] S. Okada and Sakae Fujiki, "The intrinsic substance and metrization of communication," *J. Jap. Inst. of Elec. Commun. Engrg.*, no. 204, pp. 147–157; March, 1940. (In Japanese.) English abstract: *Nippon Elec. Commun. Engrg.*, no. 21, pp. 64–65; July, 1940.

Correspondence

"In Which Fields Do We Graze?"*

I was very much interested in the above editorial because of my own views on the generality of the theory of information and also my interest as a former employee of the Federal Telecommunication Laboratories in the lead that company is taking in the administration of the PGIT.

The argument that the applications of information theory to other fields be left to specialists in those other fields is further evidence of the parochial attitude of scientists who forget that their field began as the investigation of *all* knowledge. From a study of the principles of information theory and the understanding of what constitutes *information*, a much larger view of all human endeavor unfolds, so that I realize that although other fields such as management, biology, psychology, business relationships, law, and so on, do often reach the same conclusions, as are analytically determined in our field, they do it by such a laborious method that their handicap is evident. (I include several ancillary fields such as game theory and symbolic logic as part of the science of information.)

Several examples may illustrate: Prof. Lillian Lieber of Brooklyn Polytechnic Institute and St. John's University felt that the entire Aristotelian syllogism which occupied philosophers for several thousand years could be condensed to three relationships of Boolean Algebra (symbolic logic). She also indicated that all American legal principles, when analyzed by mathematical logic, may result in such discovery of inconsistencies logical as to render a good part void.

The work done at the University of Illinois biology laboratories has indicated that the redundancy principle is not unknown in nature. Many human functions are carried out by a redundant use of nerves and muscles. This accounts for the extraordinarily high "accuracy of transmission" if we merely substitute a few biological terms for "accuracy" and "transmission." Perhaps "successful functioning" would do for the former, and "process" such as birth, grasp, lift, walk, etc. for the latter. I was particularly struck by the similarity when studying the operation of the Naval Ordnance Research Calculator formerly at Columbia University. There was this huge machine (the brain), being instrumented (a logic), fed (power) cooled to a constant temperature (not too far from 98.6 degrees F), and defective plug-in assemblies replaced on a continuous inspection basis which was so close to cellular replacement in the animal organism as to be startling! The chief engineer on the project had determined (empirically) that the best technique was to inspect regularly and replace only those plug-in assemblies which were going bad (due to loss of G_m in their tubes etc.) restoring those units which were good although some of these had been in operation much longer than statistical averages. This is precisely what occurs in living cells! The use of code checking and redundancy in the computer is analogous to the redundancy of the organism. A familiarity with information theory by the biologist could help him predict how many original cells are necessary to insure a satisfactory biological function with a given percentage of "transmission accuracy," that is, "successful functioning." In particular, I was thinking of the reason why nature may require millions of male cells per successful fertilization of an ovum when only one is needed to "transmit the information." Could this account for the minute percentage of imperfect, deformed babies at birth?

In business the applications of information theory are so numerous that I feel that those businessmen who are ignorant of it through not being communication engineers or who have not learned the principles naturally are at a distinct handicap. Statements such as, "We expect to pay a high price for it," do not convey information, and so I personally do not act upon them. The statement, "We expect to pay a higher price than last time," conveys a measurable amount of information which can be expressed in bits; and the statement, "We expect to pay no less than $8 nor more than $10," can even be instrumented on a computer together with other contract details.

The last example is one enabling me to solve many engineering problems by an extension of the elements of information from one field to another. This device is familiar to engineers who use analogs in the solution of problems, but the more extensive generalization, the concept of the propositional function of Bertrand Russell, and the algebra of classes, permit a bolder use of analogs retaining only those elements which *convey information*.

Shannon's theory is admittedly limited to the engineering aspects of communication and does not include what he calls the semantic aspects which he considers irrelevant to the engineering problem. However, an investigation into the ramifications of the science, such as, for example, music (what is it about the *transmitter*, the *channel*, and the *receiver*, which conveys *information*?) must of necessity include almost all other scientific disciplines.

This lengthy letter is prompted by the desire to do something to counteract the narrowing of that branch of our science, information theory, which seems to me to hold promise of providing the key to what makes almost everything tick, and not to restrict it merely to radio and wire communication.

Max Hoberman
Bergen Laboratories
Fair Lawn, N. J.

You bring up a very interesting question in your editorial.

Having attempted to apply Information Theory to a field somewhat removed from radio, I might have a vested interest in claiming that our field is the universe. Seriously however, I believe that since Information Theory has been developed as a new tool by our group of engineers, we can be considered duty bound to utilize the tool in every way that it may be of benefit. This is especially true, since our group includes the only nucleus of the craftsmen capable of utilizing Information Theory and *evaluating* such utilization. Thus any member attempting to extend the scope of the theory, can only do so properly within out limited group. No other place exists. For that reason, we have as little right to disown our products as to disown our physical offsprings.

Until other groups are formed into which we can extend its scope, it seems to me we are duty bound to follow every rich vein into whatever field it leads. Such a course of action would be nothing new for the IRE. The IRE has grown to include every group that used the tools initially developed for the radio industry. Thus, for instance, the electronic computer group, industrial electronics group, instrumentation group, ultrasonic engineering group, medical electronics group, and substantial portions of the other groups have "grazed into fields" that were far removed from the original concept of radio. The editor of the Proceedings of the IRE comments on that fact in the February, 1956 issue (page 151) and points out that a minority of the members are actually engaged in radio. Thus, we have both logic and precedent on the side of not limiting the investigations of the Information Theory Group.

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I am writing in response to the question "In Which Fields Do We Graze?" It is my feeling that the PGIT should be interested in the broadest possible range of applications of information theory and com-

*Trans. IRE, vol. IT-1, p. 2; December, 1955.

unication theory, both because the further evelopment and application of these fields ill thereby be promoted, and because the est and most interested audience for papers a such work is to be found among the GIT. However, care should be exercised avoid bringing papers to the attention of e PGIT merely for their novelty; they ould represent good information or communication theory. New applications should also be brought to the attention of people within the specialized field of the new applications, but the type of presentation required there will be quite different from what would suit the PGIT.

The problem would be simplified by defining "Information Theory" as the study of measures of information, regardless of the field of application. As such, information theory is a subfield within "communication theory," which may be defined as the study of the statistical characteristics of signals corrupted by noise, and which likewise has a broad range of application.

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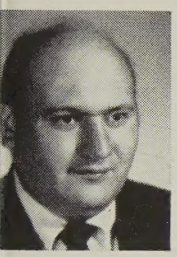
Electronic Defense Lab.,
Mountain View, Calif.

Contributors

Marvin Blum was born on June 18, 1928 New York City. He received the B.S. gree from Brooklyn College in June, 1948, and has taken graduate courses in mathematics, physics, and electrical engineering from several universities, including George Washington University, American University, Maryland University, National Bureau of Standards School, and U.C.L.A. Extension.



M. Blum

Mr. Blum worked at the National ureau of Standards in the Central Radio opagation Laboratory until 1950. He en transferred to the Ordinance Division, here he conducted radar reflection studies ating to missile proximity fuzes. Since July, 1954, Mr. Blum has been ployed at Convair, San Diego, where he conducting theoretical investigations in oothing and prediction filters, noise mulation, and data reduction.

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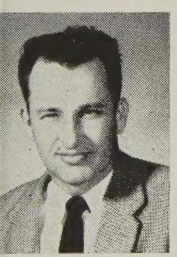
Paul E. Green, Jr. (S '46—A '48) was rn in Durham, N.C., in 1924. He received e B.A. degree in Physics from the University of North Carolina in 1944. After a brief tour of duty as a Navy electronics officer, he entered North Carolina State College, receiving the M.S.E.E. degree in 1948.



P. E. Green, Jr.

He then joined the Agricultural Engineering Department of the same institution as Research Assistant Professor to udy photoelectric color measurement d grading techniques. In 1949 he became a Research Assistant in the Research Laboratory of Electronics at M.I.T. working in statistical communication techniques. In 1951 he joined Lincoln Laboratory in the same capacity and received the Sc.D. degree from M.I.T. in 1953. Since that time he has been a staff member at Lincoln Laboratory, doing research on communication techniques.

Dr. Green is a member of Sigma Xi.

❖

Kent R. Johnson (S '47—A '55) was born in Evanston, Ill., on Aug. 26, 1929. He received the B.E.E. degree from Vanderbilt University in 1950, the M.S. degree in electrical engineering from Carnegie Institute of Technology in 1952, and the Ph.D. degree in electrical engineering from Northwestern University in 1954. While at Northwestern University he was first employed by the Microwave Laboratory of the Technological Institute, and was later a resident in research at that Institute's Communications Circuitry Laboratory.

K. R. Johnson

Upon leaving Northwestern in 1954, Dr. Johnson was employed by the Ramo-Wooldridge Corporation of Los Angeles, where he was concerned with noise and filtering problems in servomechanism design.

In 1955 he joined the faculty of the University of Pittsburgh where he is an assistant professor of electrical engineering.

Dr. Johnson is a member of Phi Kappa Phi, Tau Beta Pi, and Sigma Xi.

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J. A. McFadden was born December 11, 1924 in San Juan, P.R. He received the B.S.E. degree in mathematics in 1945 and the B.S.E. in electrical engineering in 1946, both from the University of Michigan. The same institution granted him the M.S. degree in physics in 1947 and the Ph.D. in physics in 1951. While in graduate school he served as research assistant and as teaching fellow.

J. A. McFadden

Since 1951 he has been employed as a physicist at the Naval Ordnance Laboratory, White Oak, Md., working primarily in applied mathematics. His specialties have been fluid dynamics, acoustics, and, more recently, probability, noise, and stochastic processes. He has also done part-time teaching for Montgomery Junior College and for the University of Maryland.

Dr. McFadden is a member of the American Physical Society, Tau Beta Pi, Phi Kappa Phi, and Sigma Xi.

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Kenneth S. Miller (A '47—M '52) was born on June 4, 1922 in New York City. He received the B.S. degree in chemical engineering and the A.M. and Ph.D. degrees in Mathematics from Columbia University. His post-doctoral work was done at Institute for Advanced Study, Princeton, N. J., in 1950.

K. S. Miller

During World War II Dr. Miller served as a radar officer in the U. S. Navy. Since then he has acted as a consultant to various industrial and government agencies on problems associated with system analysis, noise and applied mathematics. He is the author or co-author of numerous research

papers and five books. He is now an associate professor of mathematics at New York University.

Dr. Miller is a member of the American Mathematical Society, Sigma Xi, Tau Beta Pi, and Pi Mu Epsilon.

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Edward L. O'Neill was born in Boston, Mass., on November 29, 1927. He received the B.A. degree from Boston College in 1949, and the M.A. and Ph.D. from Boston University in 1951 and 1954.

E. L. O'Neill

From 1949 to 1950, Dr. O'Neill was a Teaching Fellow at Tufts College. He was employed part-time at the Gas Turbine Division of the General Electric Co. from 1951 to 1952. Since then he has engaged in applications of communication theory to optics, and has taught theoretical physics at Boston University.

Dr. O'Neill is a member of the American Institute of Physics, the American Association of Physics Teachers, Sigma Xi, and Sigma Pi Sigma.

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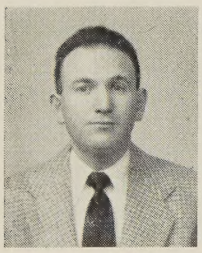
Mischa Schwartz (S '46— A '49—M '54—SM '55) was born in New York, N.Y., on September 21, 1926. He served in the Signal Corps during World War II, and returned to Cooper Union to receive the B.E.E. degree in 1947. He received the M.E.E. degree from the Polytechnic Institute of Brooklyn in 1949, and the Ph.D. degree from Harvard University in 1951, the latter under a scholarship grant from the Sperry Gyroscope Company.



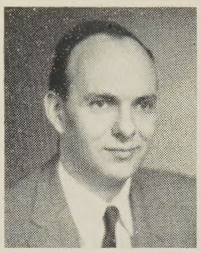
M. Schwartz

From 1947 to 1952 Dr. Schwartz was employed as a Project Engineer at the Sperry Gyroscope Company. Since 1952 he has been Assistant Professor of Electrical Engineering at the Polytechnic Institute of Brooklyn, and has at various times been a part-time Graduate Lecturer at Adelphi College and the College of the City of New York. He has also been a Radiation Physicist at Montefiore Hospital.

He is a member of Tau Beta Pi, Eta Kappa Nu, Sigma Xi, and the American Society for Engineering Education.

❖

David Slepian (A '52) was born on June 30, 1923, in Pittsburgh, Pa. He attended the University of Michigan from 1941 to 1943, at which time he entered the U.S. Army Signal Corps. In 1946 he entered Harvard University, Cambridge, Mass., where he received the M.A. degree in 1947, and the Ph.D. degree in physics in 1949. He studied in Europe for one year during 1949–1950 as a Parker Fellow from Harvard University.



D. Slepian

Dr. Slepian joined the staff of the Bell Telephone Laboratories in 1950 and has been engaged in mathematical research in the fields of communication theory, switching theory, and noise theory.

He is a member of the American Mathematical Society, American Association for the Advancement of Science, and Sigma Xi.

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Seymour Stein (S '48—A '54) was born in Brooklyn, N.Y., on April 4, 1928. He has received the B.E.E. degree at City College of New York, the S.M. degree in Applied Physics at Harvard University and the Ph.D. degree in Applied Physics at Harvard University. He was a part-time Teaching Fellow at Harvard in 1950–1951, and held an RCA Predoctoral Fellowship in Electronics, under the National Research Council, in 1951–1952 and 1952–9153.

S. Stein

From 1953–1956, Dr. Stein was engaged in systems research and studies at the Waltham Laboratories of Sylvania Electric Products, Inc. In 1955–1956, he was also an instructor in Microwaves in the evening Graduate School of Engineering at Northeastern University. In March, 1956, he joined the Technical Staff at Hycon Eastern, Inc., Cambridge, Mass., where he is presently concerned with special analysis projects in communications research. He is a member of Eta Kappa Nu, Tau Beta Pi, and Sigma Xi.

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James E. Storer (A '54) was born in Buffalo, N. Y., on October 26, 1927. He received the A.B. degree in physics in June, 1947, from Cornell University, followed by the M.S. and Ph.D. degrees in 1948 and 1951, respectively, both from the Department of Engineering Sciences and Applied Physics at Harvard University. While at Harvard, he held an Atomic Energy Commission Fellowship.

J. E. Storer

Dr. Storer was Research Fellow at the Electronics Research Laboratory, 1951–53, and Lecturer in the Division of Applied Science, Harvard, 1952–53. He is now Assistant Professor in the Division of Applied Science, doing research in Electromagnetic theory and random processes. He was awarded a Guggenheim Fellowship for 1956, and engaged in research at Stanford University.

He is a member of Sigma Xi.

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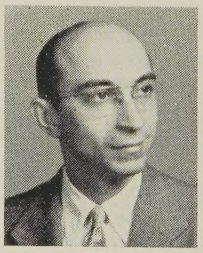
Lotfi A. Zadeh (S '45—A '47—M '50) was born on February 4, 1921, in Baku, Russia. He attended the American College of Teheran and received the B.S. degree in electrical engineering from the University of Teheran in 1942. He came to the United States in 1944 and resumed his studies at the Massachusetts Institute of Technology, receiving the M.S. degree in 1946. The same year he joined the staff of Columbia University as an instructor in electrical engineering. After receiving the Ph.D. degree from Columbia in 1949, he was promoted to assistant professorship in 1950 and to associate professorship in 1953.



L. A. Zadeh

Dr. Zadeh is a member of AIEE, American Physical Society, American Mathematical Society, Society of Industrial and Applied Mathematics, and Sigma Xi, Tau Beta Pi, and Eta Kappa Nu.

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All technical manuscripts and editorial correspondence should be addressed to Laurin G. Fischer, Federal Telecommunication Labs., 492 River Road, Nutley, N. J. Local Chapter activities and announcements, as well as other nontechnical news items, should be addressed to Nathan Marchand, Marchand Electronic Labs., Riversville Road, Greenwich, Conn.